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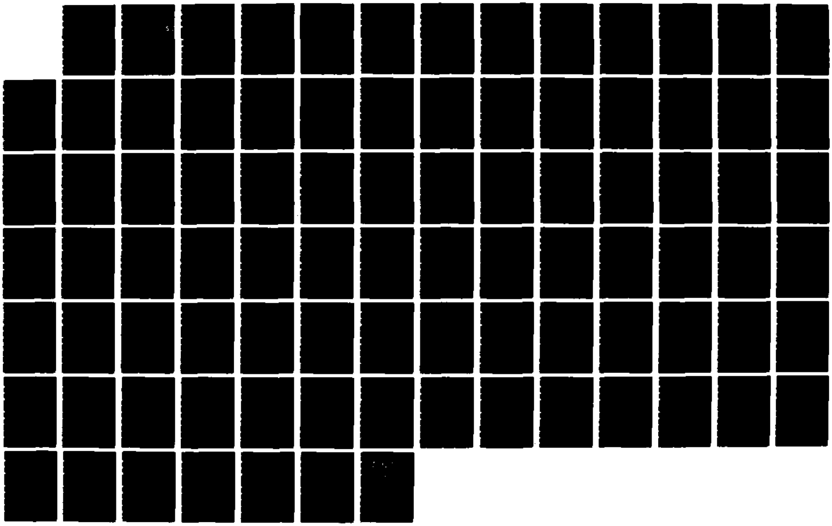

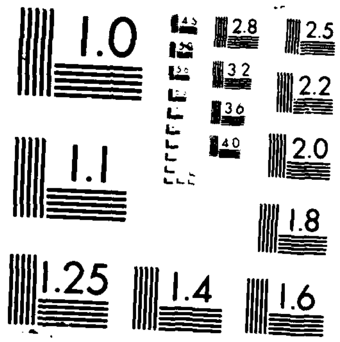

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STOCHASTIC DIFFERENTIAL EQUATIONS

DUALS OF NUCLEAR SPACES WITH SOME APPLICATIONS

by

G. KALLIANPUR

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STOCHASTIC DIFFERENTIAL EQUATIONS IN DUALS OF NUCLEAR SPACES WITH SOME APPLICATIONS

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INTRODUCTION

In these lectures I have aimed at giving an elementary introduction to certain types of stochastic differential equations in infinite dimensional spaces. The material is relatively self contained and should be easily accessible to graduate students. I have freely drawn on some of the existing work in the field, especially the pioneering work of K. Itô as well as some extensions appearing in the papers of I. Mitoma, R. Wolpert and myself, the thesis of S.K. Christensen as well as some recent work which I have been doing jointly with V. Perez-Abreu.

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LECTURE I

COUNTABLY HILBERTIAN NUCLEAR SPACES

In this lecture we introduce Countably Hilbertian Nuclear (CHN) spaces and give some examples to illustrate why these infinite dimensional spaces are convenient for the study of some practical problems, e.g. those occuring in stochastic evolutions.

Let Φ be a (real) linear space whose topology τ is given by an increasing sequence $\|\cdot\|_r$ $r \in N$ of Hilbertian norms. Let Φ_r be the Hilbert space completion of Φ w.r.t. $\|\cdot\|_r$ and assume that

$$\Phi = \bigcap_{r=1}^{\infty} \Phi_r.$$

Then (Φ, τ) is a Frechet space with metric

$$\rho(\phi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\phi - \psi\|_n}{1 + \|\phi - \psi\|_n}$$

and (Φ, ρ) is called a <u>Countably Hilbertian Space</u>. Since for $n < m$

$$\|\phi\|_n < \|\phi\|_m \qquad \phi \in \Phi$$

then

$$\Phi_m \subseteq \Phi_n \qquad m > n.$$

A countably Hilbertian space Φ is called <u>nuclear</u> if for each $n > 0$ there exists $m > n$ such that the canonical injection $i: \Phi_m \hookrightarrow \Phi_n$ is Hilbert-Schmidt i.e. if $\{\phi_j\}_{j>1}$ is a complete orthonormal system (CONS) in Φ_m then we have

$$\sum_{j=1}^{\infty} \|\phi_j\|_n^2 < \infty.$$

Let Φ_n' be the dual (Hilbert) space of Φ_n and for $f \in \Phi_n'$ let

$$\|f\|_{-n} = \sup_{\|\phi\|_n < 1} |f[\phi]|$$

since for $n < m$ $\|\phi\|_n < \|\phi\|_m$ then

$$\Phi_n' \subseteq \Phi_m' \qquad n < m.$$

Let Φ' be the topological dual space of Φ with the strong topology, which is given by the complete system of neighborhoods of zero given by sets of the form, $\{f \in \Phi' : \|f\|_B < \varepsilon\}$, $\varepsilon > 0$, where,

$$\|f\|_B = \sup\{|f[\phi]| : \phi \in B\} \quad B \text{ a bounded set in } \Phi.$$

Then

$$\Phi' = \bigcup_{n=1}^{\infty} \Phi_n'.$$

It is important to note that this topology cannot be given by a countable family of seminorms.

Suppose there is an inner product $\langle \cdot,\cdot \rangle_H$ on Φ which is continuous in the τ-topology of Φ. Let H be the Hilbert space completion of Φ w.r.t. $\langle \cdot,\cdot \rangle_H$. Then the triplet

$$\Phi \hookrightarrow H \hookrightarrow \Phi'$$

is called a <u>rigged Hilbert space</u> or a <u>Gelfand triplet</u>. The Hilbert space H may be one of the Hilbert spaces Φ_r defining the topology of Φ but this is not always the case as we shall illustrate later on.

<u>Example 1.1</u> Let $\mathcal{S}(R)$ be the space of rapidly decreasing functions on R. The usual definition of $\mathcal{S}(R)$ is the following:

$$\phi \in \mathcal{S}(R) \text{ iff } \phi \in C^{\infty} \text{ and } \sup_{x \in R} |x^{\alpha}\phi^{(\beta)}(x)| < \infty \quad \alpha, \beta \in N$$

and the topology of $\mathcal{S}(R)$ (Fréchet) is given by the family of seminorms

$$\{\|\phi\|_{\alpha,\beta} = \sup_{x \in R} |x^{\alpha}\phi^{(\beta)}(x)| \,|\, \alpha, \beta \in N\}.$$

The space $\mathcal{S}(R)$ can also be defined using the following sequence of Hilbertian norms: Let

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \left(\frac{d}{dx}\right)^n e^{-x^2/2}$$

and

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Then the sequence of Hermite functions $\{\phi_n\}_{n \geq 1}$

$$\phi_{n+1}(x) = \sqrt{g(x)}\,(n!)^{1/2} H_n(x) \qquad n \geq 0 \tag{1.1}$$

is a CONS in $L^2(R)$. Then ([5])

$$\mathcal{S}(R) = \{\phi \in L^2(R) : \|\phi\|_p^2 < \infty \quad \forall p \in R\}.$$

Let $\mathcal{S}_p$ be the completion of $\mathcal{S}(R)$ w.r.t. $\|\cdot\|_p$, then $_0 = L^2(R)$

$$\mathcal{S}_p \subset \mathcal{S}_q \qquad p > q$$

and

$$\mathcal{S}(R) = \bigcap_p \mathcal{S}_p$$

$$\mathcal{S}(R)' = \bigcup_p \mathcal{S}_p'$$

The Schwarz topology on $\mathcal{S}$ is the topology determined by $\{\|\cdot\|_p, p \in R\}$ or equivalent by the countable family $\{\|\cdot\|_p : P \in N\}$.

Let $f,g \in \mathcal{S}$, then for all $p \in R$

$$|f[g]|^2 = |\sum_n \langle f,\phi_n\rangle_0 \langle g,\phi_n\rangle_0|^2 <$$

$$(\sum_n \langle f,\phi_n\rangle_0^2 (n + \tfrac{1}{2})^{-2p})(\sum_n \langle g,\phi_n\rangle_0^2 (n + \tfrac{1}{2})^{2p}) = \|\phi\|_{-p}^2 \, \|g\|_p^2$$

which shows that $\mathcal{S}_{-p}$ is the dual of $\mathcal{S}_p$ and therefore

$$\mathcal{S}' = \bigcup_{p>0} \mathcal{S}_{-p}.$$

Finally since for any $p \in R$ $\{(n + \frac{1}{2})^{-p}\phi_n\}$ is a CONS for $_p$ then if $p > q + \frac{1}{2}$

$$\sum_{n=1}^{\infty} \|(n + \tfrac{1}{2})^{-p}\phi_n\|_q^2 = \sum_{n=1}^{\infty} (n + \tfrac{1}{2})^{-2(p-q)} < \infty$$

i.e. the canonical injection $\mathcal{S}_p \hookrightarrow \mathcal{S}_q$ is Hilbert-Schmidt for $p > q + \frac{1}{2}$.

Hence $\mathcal{S}(R)$ is a CHNS and if $H = L^2(R)$

$$\mathcal{S}(R) \hookrightarrow H \hookrightarrow \mathcal{S}(R)'$$

is a rigged Hilbert space. $\mathcal{S}(R)'$ is the familiar space of tempered distributions.

The following observation will be useful later (see Remark 1.2): If L is the operator on $H = L^2(R)$ defined by

$$-L = \frac{d^2}{dx^2} - \frac{x^2}{4}$$

then $L\phi_n = \lambda_n\phi_n$ where $\lambda_n = n - \frac{1}{2}$ and for $r > \frac{1}{2}$

$$\sum_{j=1}^{\infty} \|(I+L)^{-r}\phi_j\|_H^2 = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r} = \sum_{j=1}^{\infty} (j + \tfrac{1}{2})^{-2r} < \infty. \tag{1.2}$$

In these lectures we consider the following model for deterministic evolution:

Model. Suppose we have a rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$ on which is defined a continuous linear operator $A: \Phi \to \Phi$ and a strongly continuous semigroup $(T_t)_{t\geq 0}$ on H such that the following conditions hold:

a) $T_t \Phi \subseteq \Phi \quad \forall\, t > 0.$

b) The restriction $T_t|_\Phi : \Phi \to \Phi$ is τ-continuous $\forall\, t > 0$.

c) $t \to T_t \phi$ is continuous $\forall\, \phi \in \Phi$.

d) The generator of T_t on H coincide with A on Φ.

If Φ, H, Φ' are already given, (a)-(d) is a restriction on the type of T_t that can be considered. However, it is important to observe that in practical problems, physical considerations give no idea of the rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$ and only the Hilbert space H and the semigroup T_t are naturally given in the problem, and hence the Schwarz space cannot be chosen in advance.

The following example gives a method of choosing Φ and Φ' when T_t is given and satisfies certain conditions. Then we will introduce some examples where we can set up an infinite dimensional stochastic differential equation by choosing Φ and Φ' suitably so as to satisfy the above conditions.

Given a rigged Hilbert space $\Phi \hookrightarrow H \hookrightarrow \Phi'$, a semigroup $(T_t)_{t\geq 0}$ satisfying the above conditions will be called compatible with (Φ, H, Φ') or equivalently we will refer to (Φ, H, T_t) as a compatible family.

EXAMPLE 1.2 (A class of examples of (Φ, H, Φ', T_t)). Let H be a real separable Hilbert space and $A = -L$ be a closed densely defined self adjoint operator on H s.t. $\langle -L\phi, \phi\rangle_H \leq 0$ for $\phi \in \mathrm{Dom}(L)$. Let T_t be the semigroup on H determined by A. Further assume that some power of the resolvent of L is a

Hilbert-Schmidt operator i.e.

$$\exists \quad r_1 \text{ s.t } (\lambda I + L)^{-r_1} \text{ is Hilbert-Schmidt} \tag{1.3}$$

This condition helps to find an appropriate CHNS Φ for the model, as we shall now indicate: it implies that there is a CONS $\{\phi_j\}_{j\geq 1}$ in H s.t.

$$L\phi_j = \lambda_j \phi_j \quad \forall\ j \geq 1$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Take $\lambda = 1$ and define

$$\Phi = \{\phi \in H: \|(I + L)^r \phi\|_H^2 < \infty \quad \forall r \in R\}$$

$$= \{\phi \in H: \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle\phi,\phi_j\rangle_H^2 < \infty \quad \forall\ r \in R\} \tag{1.4}$$

Define the inner product $\langle\cdot,\cdot\rangle_r$ on Φ by

$$\langle\phi,\psi\rangle_r := \sum_{j=1}^{\infty} (1 + \lambda_j)^{2r} \langle\phi,\phi_j\rangle_H \langle\psi,\phi_j\rangle_H \tag{1.5}$$

and

$$\|\phi\|_r^2 = \langle\phi,\phi\rangle_r .$$

Let Φ_r be the $\|\cdot\|_r$ - completion of Φ. We then have

$$\Phi = \bigcap_r \Phi_r, \qquad \Phi' = \bigcup_r \Phi_r'$$

and for $r \leq s$, $\|\phi\|_r \leq \|\phi\|_s$ and so $\Phi_s \subseteq \Phi_r$ with $\Phi_0 = H$. Condition (1.3) implies that the canonical injection $\Phi_\rho \hookrightarrow \Phi_r$ is Hilbert Schmidt for $\rho > r + r_1$ and therefore Φ is a CHNS.

For each $r > 0$ Φ_{-r} and Φ_r are in duality under the pairing

$$\eta[\phi] := \sum_{j=1}^{\infty} \langle\eta,\phi_j\rangle_{-r} \langle\phi,\phi_j\rangle_r \quad \eta \in \Phi_{-r}, \quad \phi \in \Phi_r \tag{1.6}$$

and therefore $\Phi_{-r} = \Phi_r'$. We also have that $\{\phi_j\}_{j\geq 1}$ is a complete orthogonal system (not normal) in Φ_r for all $r \in R$.

From now on we will write $\langle\cdot,\cdot\rangle_0 = \langle\cdot,\cdot\rangle_H$ and

$$\theta_1 = \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty . \tag{1.7}$$

Now we shall prove that the semigroup T_t satisfies conditions (a)-(d) for our above model for deterministic evolution:

Condition (1.3) implies that $-L$ generates a contraction semigroup T_t on H. For $\phi \in \Phi$ and $t > 0$ we have

$$T_t\phi = \sum_{j=1}^{\infty} e^{-t\lambda_j}\langle\phi,\phi_j\rangle_0\phi_j \in \Phi$$

which implies (a). Next for $t > 0$ and $\phi \in \Phi$

$$\|T_t\phi\|_r^2 = \sum_{j=1}^{\infty} e^{-2t\lambda_j}(1 + \lambda_j)^{2r} \langle\phi,\phi_j\rangle_0^2 \leq \sum_{j=1}^{\infty} (1+\lambda_j)^{2r}\langle\phi,\phi_j\rangle_0^2 = \|\phi\|_r^2 .$$

Then since $\psi_n \to 0$ in Φ if and only if $\|\psi_n\|_r \to 0$ $\forall\, r \in R$,

$$\|T_t\psi_n\|_r \to 0 \quad \forall\, r \in R \Rightarrow T_t\psi_n \to 0 \quad \text{in } \Phi$$

and therefore condition (b) is satisfied.

Now for $s,t \in T$ and $\phi \in \Phi$

$$\|T_t\phi - T_s\phi\|^2 = \sum_{j=1}^{\infty} (e^{-t\lambda_j} - e^{-s\lambda_j})^2 \langle\phi,\phi_j\rangle_0^2(1 + \lambda_j)^{2r}$$

and for each $j \geq 1$

$$(e^{-t\lambda_j} - e^{-s\lambda_j})^2\langle\phi,\phi_j\rangle_0^2(1+\lambda_j)^2 \leq 4\langle\phi,\phi_j\rangle_0^2(1+\lambda_j)^{2r}.$$

Since $e^{-t\lambda_j}$ is continuous on t for all $j \geq 1$ then by the Dominated convergence theorem

$$\lim_{t \to s} \| T_t \phi - T_s \phi \|_r \to 0 \quad \forall \phi \in \Phi, \quad r \in R$$

which implies (c).

Now to prove (d) let $\phi \in \Phi$ and define $\psi_n = \sum_{j=1}^{n} \langle \phi, \phi_j \rangle_0 \phi_j$. Then $\psi_n \to \phi$ on Φ,

$$-L\psi_n = \sum_{j=1}^{n} \langle \phi, \phi_j \rangle_0 L\phi_j = -\sum_{j=1}^{n} \lambda_j \langle \phi, \phi_j \rangle_0 \phi_j$$

and for $m > n$ and $r \in R$

$$\Big\| \sum_{j=n+1}^{m} -\lambda_j \langle \phi, \phi_j \rangle_0 \phi_j \Big\|_r^2 = \sum_{j=n+1}^{\infty} \lambda^2 (1 + \lambda_j)^{2r} \langle \phi, \phi_j \rangle_0^2 \to 0 \quad \text{as} \quad n,m \to \infty.$$

Hence if $\psi = -\sum_{j=1}^{\infty} \lambda_j \langle \phi, \phi_j \rangle_0 \phi_j$ we have $-L\psi_n \to \psi$ on Φ. But since $-L$ is closed in H and $\| \cdot \|_H$ is Φ-continuous, then $\phi \in \mathrm{Dom}(L)$ and $\psi = -L\phi$ $\phi \in \Phi$, i.e.

$$-L\phi = -\sum_{j=1}^{\infty} \lambda_j \langle \phi, \phi_j \rangle_0 \phi_j$$

and

$$\| -L\phi \|_r^2 < \| \phi \|_{r+1}^2 \quad \forall \phi \in \Phi \quad r \in R.$$

Hence, $-L\Phi \subseteq \Phi$ and $-L$ is Φ-continuous which implies (d).

<u>Remarks</u>

<u>1.1</u>. A compatible family (Φ, H, T_t) or (Φ, H, L) is called a <u>special compatible family</u> if the generator L satisfies condition (1.3) and Φ is constructed as in Example 1.2, i.e. Φ is given by (1.4).

<u>1.2</u>. The Schwarz space $\mathcal{S}(R)$ of Example 1.1 may be obtained in the framework of the last example by taking

$$-L = \frac{d^2}{dx^2} - \frac{x^2}{4}, \quad H = L^2(R), \quad \lambda_j = j - \frac{1}{2} \quad j \geq 1$$

and $\{\phi_j\}$ the Hermite functionals given by (1.1). Then from (1.2) we have that $-L$ satisfies condition (1.3) for $r_1 > \frac{1}{2}$ and $(\mathcal{S}(R), L^2(R), L)$ is a special compatible family.

The following example occurs in core conductor theory and will be later considered in connection with applications to neuronal behavior.

EXAMPLE 1.3 Let $H = L^2([0,b],dx)$ and consider the deterministic problem

$$\begin{aligned} &\frac{\partial V}{\partial t} = -\alpha V + \beta \Delta V \quad t > 0 \quad 0 < x < b, \\ &\qquad\qquad (\alpha, \beta > 0 \ \text{constants}), \\ &V(0,x) = v_0(x), \qquad\qquad\qquad\qquad (\mathrm{I}) \\ &\frac{\partial V}{\partial x}(t,0) = \frac{\partial V}{\partial x}(t,b) = 0. \end{aligned}$$

Let (T_t) be the continuous contraction semigroup on H defined by (I), i.e.

$$(T_t f)(x) = \int_0^b G(x,y;t) f(y) dy \qquad f \in H$$

where

$$G(x,y;t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y)$$

$$\lambda_n = \alpha + \beta\left(\frac{n\pi}{b}\right)^2 \quad n \geq 1, \tag{1.8}$$

and

$$\phi_n(x) = \left(\frac{2}{b}\right)^{1/2} \cos\left(\frac{n\pi x}{b}\right) \quad n \geq 1. \tag{1.9}$$

Let $A = -L$ be the generator of T_t. Then (I) has the solution

$$V(t,x) = \int_0^b G(x,y;t)v_0(y)dy \quad , \; t > 0 \tag{1.10}$$

for $v_0 \in \mathrm{Dom}(L)$.

Now consider the stochastic model

$$\frac{\partial V}{\partial t} = -\alpha V + \beta \Delta V + \dot{W}_{t,x} \tag{II}$$

where $\dot{W}_{t,x}$ is a Gaussian white noise in space time. We shall show how to set up model II as a Φ'-valued stochastic differential equation for a suitable definition of Φ.

For λ_n and ϕ_n as in (1.8) and (1.9) respectively and

$$L = -\alpha I + \beta\Delta \qquad (\Delta = \frac{d^2}{dx^2})$$

define Φ by the method explained in Example 1.2 i.e.

$$\Phi = \{\phi \in H : \sum_{j=1}^{\infty} (1 + \lambda_j)^r \langle\phi,\phi_j\rangle_0^2 < \infty \quad \forall r \in R\}$$

where

$$\langle\phi,\psi\rangle_0 = \int_0^b \phi(x)\psi(x)dx.$$

If $p > m + \frac{1}{4}$, since $\{(1 + \alpha + \beta(\frac{n\pi}{b})^2)^{-p}\phi_n\}$ is a CONS for Φ_p,

$$\sum_{n=1}^{\infty} \|(1 + \alpha + \beta(\frac{n\pi}{b})^{-p}\phi_n\|_m^2 = \sum_{n=1}^{\infty} (1 + \alpha + \beta(\frac{n\pi}{b}))^{-2(p-m)} < \infty$$

and then the injection map $\Phi_p \hookrightarrow \Phi_m$ is Hilbert Schmidt for $p > m + \frac{1}{4}$.

Hence Φ is a CHNS and is the linear space of all infinite differentiable real functions f such that $f^{(k)}(0) = f^{(k)}(b) = 0$ for all $k = 0,1,2,\ldots$.

Furthermore

a) $T_t \Phi \subseteq \Phi$

b) $L\Phi \subseteq \Phi$

c) $L|_{\Phi} = (-\alpha I + \beta\Delta)$

and the operator $(\lambda I + L)^{-r_1}$ is Hilbert-Schmidt if $r_1 > \frac{1}{4}$:

$$\sum_{j=1}^{\infty} \|(\lambda I + L)^{-r_1}\phi_j\|_0^2 = \sum_{j=1}^{\infty} (\lambda + \alpha + \beta(\frac{j\pi}{b}))^{-2r_1} < \infty$$

Now we indicate how to set up model II as a Φ'-valued stochastic differential equation. We proceed heuristically. From (1.10) we have that (II) can be written as

$$\frac{d}{dt} T_t v_0 = -T_t L v_0 + \dot{W}_{t,x} .$$

Define for $t > 0$

$$\xi_t[\phi] = \int_0^b V(t,x)\phi(x)dx \qquad \phi \in \Phi ,$$

then for $\phi \in \Phi$

$$\frac{d}{dt} \xi_t[\phi] = \int_0^b \frac{\partial V}{\partial t} \phi(x)dx$$

$$= -\int_0^b LV(t,x)\phi(x)dx + \int_0^b \phi(x)\dot{W}_{t,x}dx$$

i.e.

$$d\xi_t[\phi] = -\xi_t[L\phi]dt + dW_t[\phi] \tag{1.11}$$

where

$$W_t[\phi] := \int_0^b \phi(x)\ \dot{W}_{t,x}dx, \tag{1.12}$$

$$\xi_0[\phi] = \int_0^b v_0(x)\phi(x)dx$$

and

$$EW_t[\phi]W_s[\psi] = \min(t,s)\langle\phi,\psi\rangle_H \quad \text{for} \quad \phi,\psi \in \Phi. \tag{1.13}$$

If on Φ' we define T_t' by $T_t'f[\phi] = f[T_t\phi]$ for $f \in \Phi'$ then $L'f[\phi] = f[L\phi]$ and one may write model II as

$$d\xi_t = -L'\xi_t dt + dW_t$$

and (II)'

$$\xi_0 \text{ given as above.}$$

where W_t is a Φ'-valued Wiener process with covariance $\langle\cdot,\cdot\rangle_H$ as defined in the next section, and a special case of a Φ'-martingale.

Model (II)' is an example of a Φ'-valued linear stochastic differential equation for which we want to solve $\xi_t \in \Phi'$ and

$$\xi \in C(R_+;\Phi')$$

where $C(R_+;\Phi')$ is the space of all Φ'-valued processes on R with continuous paths in the strong topology of Φ'.

LECTURE II

MARTINGALES TAKING VALUES IN DUALS OF NUCLEAR SPACES

Throughout this lecture we assume that $(\Omega, \mathfrak{F}, P)$ is a complete probability space with a right continuous filtration $(\mathfrak{F}_t)_{t\geq 0}$. Also, unless otherwise explicitly stated, we assume that Φ is a given Countably Hilbertian nuclear space defined by a chain of Hilbert spaces $\Phi_p \subseteq \Phi_q$ $q < p$ with norms $\|\cdot\|_p \geq \|\cdot\|_q$ and strong duals $\Phi'_q \subseteq \Phi'_p$. We denote by Φ' the strong dual of Φ and by $\mathcal{B} = \mathcal{B}(\Phi')$ the σ-field of Φ' generated by the sets

$$\{f \in \Phi' : f[\phi] < a\} \quad \phi \in \Phi \quad a \in R.$$

Definition 2.1 A mapping $X_t(\omega)$: $R_+ \times \Omega \to \Phi'$ is a Φ'-process if $X_t(\omega)[\phi]$ is a real valued process $\forall\ \phi \in \Phi$, i.e.

$$\{w: X_t(\omega) \in B\} \in \mathfrak{F} \quad \forall \quad B \in \mathcal{B}.$$

In this lecture we introduce two special cases of Φ'-valued processes, namely the Φ'-Wiener process and Φ'-martingales. We give several examples of Wiener processes and illustrate how some infinite dimensional extensions of the real valued Brownian motion (as the cylindrical Brownian motion and a sequence of independent Brownian motions) may be seen as nuclear space valued Wiener processes.

In the case of Φ'-martingales we will make the assumption that $EX_t[\phi]^2 < \infty$ $\forall \phi \in \Phi$, $t \geq 0$. This condition is not necessary and will be assumed only for simplicity in order to show the kind of techniques used in the study of Φ'processes. Examples of these techniques are the following two lemmas. They will be referred in the future as the regularization Theorem and the Baire category argument.

Lemma 2.1 (Regularization) (Itô) Let $Y: \Phi \to L^2(\Omega, \mathfrak{F}, P)$ be a continuous linear map. Then there exists a Φ'-valued random variable $\tilde{Y}$ s.t.

$$\tilde{Y}(\omega)[\phi] = Y(\phi)(\omega) \quad P \text{ a.s. } \forall \phi \in \Phi.$$

Moreover there is a $q > 0$ s.t. $P(\tilde{Y} \in \Phi_q') = 1$.

Proof. Let $V(\phi) = E(Y(\phi)^2)$ $\forall \phi \in \Phi$. Since V is continuous there exists $r > 0$ and $\delta > 0$ s.t. if $\|\phi\|_r < \delta$ then $V(\phi) < 1$. Hence if $\theta = 1/\delta$ we have

$$V(\phi) < \theta^2 \|\phi\|_r^2 \quad \forall \phi \in \Phi \quad . \tag{2.1}$$

Let $q > r$ be such that the injection mapping $\Phi_q \hookrightarrow \Phi_r$ is Hilbert-Schmidt and let $\{\phi_j\}_{j \geq 1}$ Φ be a CONS for Φ_q. Then from (2.1)

$$E\left(\sum_{j=1}^{\infty} Y(\phi_j)^2\right) < \theta^2 \sum_{j=1}^{\infty} \|\phi_j\|_r^2 < \infty$$

i.e. if $\Omega_1 = \{\omega: \sum_{j=1}^{\infty} (Y(\phi_j)(\omega))^2 < \infty\}$ then $P(\Omega_1) = 1$. Define

$$Y(\omega) = \begin{cases} \sum_{j=1}^{\infty} Y(\phi_j)(\omega)\tilde{\phi}_j & \text{if } w \in \Omega_1 \\ 0 & \text{otherwise} \end{cases}$$

where $\{\tilde{\phi}_j\}_{j \geq 1}$ is the CONS of Φ_q' dual to $\{\phi_j\}_{j \geq 1}$. Then $\tilde{Y}$ is a Φ'-valued random variable s.t. $Y \in \Phi_q'$ a.s. and

$$\tilde{Y}(\omega)[\phi] = \sum_{j=1}^{\infty} Y(\phi_j)(\omega) \langle \phi, \phi_j \rangle_q \quad \phi \in \Phi \text{ a.s.} \tag{2.2}$$

Letting $\psi_n := \sum_{j=1}^{n} \langle \phi, \phi_j \rangle_q \phi_j$, $\|\psi_n - \phi\|_r < \|\psi_n - \phi\|_q \xrightarrow[n \to \infty]{} 0$ so that from (2.1)

$$E[\sum_{j=1}^{n} Y(\phi_j)\langle\phi,\phi_j\rangle_q - Y(\phi)]^2 \to 0 \quad \text{as} \quad n \to \infty \tag{2.3}$$

Finally, from (2.2) and (2.3) we have

$$E[\tilde{Y}(\omega)[\phi] - Y(\phi)(\omega)]^2 = 0$$

i.e.

$$\tilde{Y}[\phi] = Y(\phi) \quad \text{a.s.} \quad \forall\ \phi \in \Phi$$

Q.E.D.

From the proof of the above lemma we obtain the following result.

Corollary 2.1. Let $Y: \Phi \to L^2(\Omega, \mathfrak{F}, P)$ be a continuous linear map. Then there exist $\theta > 0$ and $r > 0$ s.t.

$$E(Y(\phi))^2 \le \theta^2 \|\phi\|_r^2 \quad \forall\ \phi \in \Phi .$$

Lemma 2.2 Let $V(\cdot): \Phi \to [0,\infty)$ satisfy the following conditions:

(1) V is lower semicontinuous, i.e. $\phi_n \to \phi \Rightarrow V(\phi) \le \underline{\lim}\ V(\phi_n)$

(2) $V(\phi + \psi) \le V(\phi) + V(\psi) \qquad \forall\ \phi,\psi \in \Phi$

(3) $V(a\phi) = |a|v(\phi) \qquad \forall\ a \in R,\ \phi \in \Phi$

(4) $V(\phi) < \infty \qquad \forall\ \phi \in \Phi$

Then $V(\phi)$ is a continuous function in ϕ and there exist $\theta > 0$ and $r > 0$ s.t.

$$V(\phi) \le \theta\|\phi\|_r \quad \forall\ \phi \in \Phi$$

Proof.

Let $D_n = \{\phi \in \Phi: V(\phi) \le n\}$

Since V is a lower semicontinuous function on Φ, then for each $n \ge 1$ D_n is a closed set of Φ. Condition 4) implies

$$\Phi = \bigcup_{n=1}^{\infty} D_n \tag{2.4}$$

Then by the Baire category theorem, since Φ is a complete metric space, it is never the union of a countable number of nowhere dense sets. Therefore there exist $\phi_0 \in \Phi$, $\delta_1 > 0$ and a positive integer r such that

$$\mathcal{U}_{\phi_0} = \{\phi \in \Phi: \|\phi - \phi_0\|_r < \delta_1\} \subseteq D_{n_0}.$$

Hence, for any $\phi \in \Phi$, $\phi \neq 0$ if $\delta < \delta_1$

$$\delta \frac{\phi}{\|\phi\|_r} + \phi_0 \in u_{\phi_0}$$

and

$$\phi_0 - \delta \frac{\phi}{\|\phi\|_r} \in u_{\phi_0}$$

and therefore they belong to D_{n_0}, i.e.

$$V(\delta \frac{\phi}{\|\phi\|_r} + \phi_0) < n_0 \tag{2.5}$$

and

$$V(\phi_0 - \delta \frac{\phi}{\|\phi\|_r}) < n_0 \tag{2.6}$$

But using 3) with $a = -1$

$$V(\delta \frac{\phi}{\|\phi\|_r} - \phi_0) = V(\phi_0 - \frac{\delta\phi}{\|\phi\|_r}).$$

Then by 2)

$$V(2\delta \frac{\phi}{\|\phi\|}) < V(\delta \frac{\phi}{\|\phi\|_r} + \phi_0) + V(\delta \frac{\phi}{\|\phi\|_r} - \phi_0) < 2n_0$$

and hence using 3), if $\theta = n_0/\delta$

$$V(\phi) \leq \theta\|\phi\|_r \quad \forall\phi \in \Phi$$

Then the continuity of V follows since using 2)

$$|V(\phi) - V(\psi)| \leq V(\phi - \psi) \leq \theta\|\phi - \psi\|_r \quad \phi,\psi \in \Phi.$$

Q.E.D.

As an example of a typical application of the above lemma (Baire category argument) we have the following result.

Corollary 2.2. Let $(X_t)_{t\geq 0}$ be a Φ'-valued stochastic process s.t.

$$EX_t^2[\phi] < \infty \quad \forall\phi \in \Phi, \quad t \geq 0.$$

Then for each $t \geq 0$ there exist $\theta_t > 0$ and $r_t > 0$ s.t.

$$E(X_t)[\phi])^2 \leq \theta_t^2\|\phi\|_{r_t}^2 \quad \forall\phi \in \Phi.$$

Proof. For each $t \geq 0$ define

$$V_t(\phi) = \{E(X_t[\phi])^2\}^{1/2} \quad \phi \in \Phi$$

Applying Fatou's lemma we have that $V_t(\phi)$ is a lower semicontinuous function on Φ. Then since V_t satisfies properties (2)-(4) in Lemma 2.2 the corollary follows applying the named lemma.

Q.E.D.

A result of this type, involving a Baire category argument, was first used in the study of Φ'-valued stochastic processes in Mitoma [10].

WIENER PROCESS TAKING VALUES IN THE DUAL OF A NUCLEAR SPACE

Definition 2.2 A strongly sample continuous Φ'-valued stochastic process $W = (W_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, P)$ is called a centered Φ'-Wiener process with covariance $Q(\cdot,\cdot)$ if W_t satisfies the following three conditions:

a) $W_0 = 0$ a.s.

b) W_t has independent increments, i.e. the random variables

$$W_{t_1}[\phi_1],\ (W_{t_2} - W_{t_1})[\phi_2],\ldots,(W_{t_n} - W_{t_{n-1}})[\phi_n]$$

are independent $\forall\ \phi_1,\ldots,\phi_n \in \Phi,\ 0 < t_1 < t_2 < \ldots < t_n,\ n > 1.$

c) For each $t > 0$ and $\phi \in \Phi$

$$E(e^{iW_t[\phi]}) = e^{-1/2(tQ(\phi,\phi))} \tag{2.7}$$

where Q is a covariance functional, i.e. a positive definite symmetric bilinear form continuous on $\Phi \times \Phi$.

Remarks

2.1. Let $(W_t)_{t>0}$ be a Φ'-Wiener process with covariance Q. Then

$$\{W_t[\phi],\ \phi \in \Phi,\ t > 0\}$$

is a centered Gaussian system and

$$E(W_t[\phi]W_s[\psi]) = \min(s,t)Q(\phi,\psi) \quad \phi,\psi \in \Phi, \quad s,t > 0. \tag{2.8}$$

2.2 A Φ'-valued process $(Z_t)_{t>0}$ is a Non-centered Wiener process if there exists $m \in \Phi'$ s.t. $Z_t - tm$ is a centered Wiener process.

We now prove the existence of a Φ'-valued Wiener process following Perez-Abreau [13].

Theorem 2.1 Let $\{Y(t,\phi),\ \phi \in \Phi,\ t > 0\}$ be a centered Gaussian system of random variables s.t.

$$E(Y(t,\phi)Y(s,\psi)) = \min(t,s)Q(\phi,\psi) \tag{2.9}$$

where Q is a covariance functional on $\Phi \times \Phi$. Then the following is true:

1). There exists a centered Φ'-valued Wiener process $W = (W_t)_{t \geq 0}$ with covariance functional Q s.t.

$$Y(t,\phi) = W_t[\phi] \quad \text{a.s.} \quad \forall \phi \in \Phi, \ t \geq 0$$

2). There exists $q > 0$ depending only on Q and not on t s.t.

$$W_\cdot \in C(R_+, \Phi_q') \quad \text{a.s.}$$

where $C(R_+, \Phi_q')$ is the space of strongly continuous functions from R_+ to Φ_q'.

Proof. Letting $V^2(\phi) = Q(\phi,\phi)$ $\phi \in \Phi$, V is a function that satisfies conditions (1)-(4) of Lemma 2.2. Then there exist $\theta > 0$ and $r > 0$ s.t.

$$Q(\phi,\phi) \leq \theta \|\phi\|_r^2 \quad \forall \phi \in \Phi. \tag{2.10}$$

By the nuclearity of Φ there exists $q > r$ s.t. the injection $\Phi_q \hookrightarrow \Phi_r$ is a Hilbert-Schmidt map i.e. if $\{\phi_j\}_{j \geq 1} \subseteq \Phi$ is a CONS for Φ_q then

$$\sum_{j=1}^{\infty} \|\phi_j\|_r^2 < \infty. \tag{2.11}$$

Hence by (2.9), (2.10), (2.11) and the monotone convergence theorem we have

$$E\left(\sum_{j=1}^{\infty} Y(t,\phi_j)^2\right) = t \sum_{j=1}^{\infty} Q(\phi_j,\phi_j) \leq t\theta \sum_{j=1}^{\infty} \|\phi_j\|^2 < \infty.$$

Thus if $\Omega_t = \{\omega \in \Omega: \sum_{j=1}^{\infty} Y(t,\phi_j)^2 < \infty\}$ then $P(\Omega_t) = 1$.

Let $\{\hat{\phi}_j\}_{j \geq 1}$ be the CONS of Φ_q' dual to $\{\phi_j\}_{j \geq 1}$ and define

$$\widetilde{W}_t(\omega) = \begin{cases} \sum_{j=1}^{\infty} Y(t,\phi_j)(\omega)\Phi_j & \omega \in \Omega_t \\ 0 & \text{otherwise.} \end{cases}$$

Then $\widetilde{W}_t \in \Phi_q'$ a.s. for all $t > 0$,

$$\widetilde{W}_t(\omega)[\phi_j] = Y(t,\phi_j)(\omega) \quad \text{for} \quad \omega \in \Omega_t$$

and

$$\widetilde{W}_t(\omega)[\phi] = \sum_{j=1}^{\infty} \widetilde{W}_t(\omega)[\phi_j]\Phi_j[\phi] \tag{2.12}$$

For $\phi \in \Phi_q$, $\phi = \sum_{j=1}^{\infty} \langle\phi,\phi_j\rangle_q \phi_j = \sum_{j=1}^{\infty} \Phi_j[\phi]\phi_j$ it follows from (2.10) and

$$\|\phi\|_r \le \|\phi\|_q \qquad r < q$$

that

$$E(Y(t,\phi) - \sum_{j=1}^{n} \Phi_j[\phi]Y(t,\phi_j))^2 = tQ(\phi - \sum_{j=1}^{n} \Phi_j[\phi]\phi_j;\ \phi - \sum_{j=1}^{n} \Phi_j[\phi]\phi_j)$$

$$\le t\theta\|\phi - \sum_{j=1}^{n} \Phi_j[\phi]\phi_j\|_q^2 \underset{n\to\infty}{\to} 0.$$

Hence, from this and (2.12) we have

$$\widetilde{W}_t[\phi] = Y(t,\phi) \quad \text{a.s.} \quad \forall\phi \in \Phi, \quad t > 0.$$

It remains to show that $\widetilde{W}$ has a strongly continuous version.

We recall that if G is a real valued Gaussian random variable with zero mean and variance σ^2, then

$$E(G^4) = 3\sigma^4$$

Then writing $X_t^j = Y(t,\phi_j)$

$$E(X_t^j - X_s^j)^4 = 3|t - s|^2\{Q(\phi_j,\phi_j)\}^2. \tag{2.13}$$

Hence, using Holder's inequality, (2.13), (2.10) and (2.11) we have

$$\begin{aligned}
E\|\tilde{W}_t - \tilde{W}_s\|_{-q}^4 &= E\Big(\sum_{j=1}^{\infty} (X_t^j - X_s^j)^2\Big)^2 \\
&= E\Big(\sum_{j=1}^{\infty} (X_t^j - X_s^j)^4 + 2\sum_{j<k} (X_t^j - X_s^j)^2(X_t^k - X_s^k)^2\Big) \\
&\leq \sum_{j=1}^{\infty} E(X^j - X^j)^4 + 2\sum_{j<k} (E(X_t^j - X_s^j)^4 E(X_t^k - X_s^k)^4)^{1/2} \\
&= 3|t - s|^2\Big\{\sum_{j=1}^{\infty} Q(\phi_j,\phi_j)^2 + 2\sum_{j<k} Q(\phi_j;\phi_j)Q(\phi_k,\phi_k)\Big\} \\
&= |t - s|^2\Big\{\sum_{j=1}^{\infty} Q(\phi_j,\phi_j)\Big\}^2 \equiv K^2|t - s|^2
\end{aligned}$$

i.e.

$$E\|\tilde{W}_t - \tilde{W}_s\|_{-q}^4 \leq K^2|t - s|^2, \quad K \text{ constant } s,t \in R_+ \;. \tag{2.14}$$

Then the proof of the theorem is completed using the following variant, given in Itô [6], of Kolmogorov's theorem on the existence of continuous versions for stochastic processes.

Theorem 2.2 (Itô [6]). Let X be a Φ_q'-valued process. If there exist positive constants α,β,K s.t.

$$E(\|X_t - X_s\|_{-q}^{\alpha}) \leq K|t - s|^{1+\beta}$$

then X has a continuous Φ_q'-version.

In the case of a special compatible family the index q in (2) of Theorem 2.1 can be chosen in the following manner.

Corollary 2.3 Let (Φ,H,T_t) be a special compatible family and assume the hypotheses of Theorem 2.1. Then there exists a Φ'-valued Wiener process $W = (W_t)_{t\geq 0}$ with covariance functional Q such that

$$W_{\cdot} \in C(R_+;\ \Phi'_q) \text{ a.s.}$$

for any $q > r_1 + r_2$ where r_1 is given by (1.3) in Example (1.2) and r_2 is such that

$$Q(\phi,\phi) \leq \theta \|\phi\|_{r_2}^2 \quad \forall\ \phi \in \Phi$$

for some $\theta > 0$.

Proof

Form the proof of Theorem 2.1, writing r_2 instead of r in (2.10), q was taken such that the injection $\Phi_q \hookrightarrow \Phi_{r_2}$ is a Hilber-Schmidt map. But in the case of a special compatible family (see Example 1.2) the injection $\Phi_q \quad \Phi_{r_2}$ is Hilbert-Schmidt for $q > r_1 + r_2$.

Q.E.D.

Some examples of Φ'-valued Wiener processes are now introduced.

EXAMPLE 2.1. Let (Φ,H,L) be a special compatible family (see Example 1.2 and Remark 1.1). Recall that (see (1.7)) there exists $r_1 > 0$ s.t.

$$\theta_1 := \sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty$$

and the injection $\Phi_q \hookrightarrow \Phi_r$ is a Hilbert-Schmidt map for $q > r + r_1$. Let $\langle\cdot,\cdot\rangle_0$ be the inner product in H and define

$$Q_0(\phi,\psi) = \langle\phi,\psi\rangle_0 \qquad \phi,\psi \in \Phi\ .$$

Then from Theorem 2.1 there exists a Φ'-valued Wiener process $(W_t)_{t\geq 0}$ with covariance functional Q_0 s.t.

$$W \in C(R_+;\Phi_q') \text{ a.s} \quad \text{if } q > r_1$$

and will be called a Standard Wiener process.

More generally, if $r > 0$ and

$$Q_r(\phi,\psi) := \langle\phi,\psi\rangle_r \quad \phi,\psi \in \Phi$$

then there exists a Φ'-valued process $W \in C(R_+: \Phi_q')$ for $q > r + r_1$.

As will be shown in later examples, in applications the Q is not always given by one of the inner products on the Hilbert spaces defining Φ. Nevertheless since Q is continuous on $\Phi \times \Phi$, then, as in the proof of Theorem 2.1, there exist $\theta > 0$ and $r > 0$ s.t.

$$Q(\phi,\phi) \leq \theta\|\phi\|_r^2 \quad \forall\phi \in \Phi$$

and therefore there exists a Φ'-valued Wiener process W s.t.

$$W \in C(R_+;\Phi_q') \quad \text{for } q > r + r_1.$$

EXAMPLE 2.2. Let $\mathcal{S}(R)$ be the Schwarz space of Example 1.1 (see also Remark 1.2). Then $(\mathcal{S}, L^2(R), -d^2/dx^2 + x^2/4)$ is a special compatible family where $(\phi_j)_{j\geq 1}$ are the Hermite functionals given by (1.1), $\lambda_j = j - 1/2$, $j \geq 1$, $\langle\cdot,\cdot\rangle_0$ is the inner product on $L^2(R)$ and $r_1 > 1/2$. Taking $\Phi = \mathcal{S}(R)$ and $H = L^2(R)$ in the last example, from (1.2) we have that if $Q_0(\phi,\psi) = \langle\phi,\psi\rangle_0$ then the standard Wiener process $W = (W_t)_{t\geq 0}$ in $\mathcal{S}(R)'$ is such that $W \in C(R_+;\mathcal{S}_q')$ for $q > 1/2$. Clearly, there is no smallest q such that this happens.

For $\phi \in \Phi$ define

$$W_t^{(1)}[\phi] = W_t[D^2\phi] \quad \text{where } D = \frac{d}{dx}.$$

Then the covariance functional of the Φ'-valued Wiener process $W^{(1)} = (W_t^{(1)})_{t\geq 0}$

is

$$Q^{(1)}(\phi,\psi) = Q_0(D^2\phi, D^2\psi) = \langle D^2\phi, D^2\psi\rangle_0 .$$

We shall show that $W_t^{(1)} \in C(R_+;\ '_q)$ for $q > 3/2$. In general we will prove the following: Let $Q_p(\phi,\psi) = \langle\phi,\psi\rangle_p$ $\phi,\psi \in \Phi$ $p > 0$, and let $W = (W_t)_{t>0}$ be the corresponding Φ'-valued Wiener process. Define

$$W_t^{(1)}[\phi] = W_t[D^2\phi] \tag{2.15}$$

then $W^{(1)}$ is a Φ'-valued Wiener process s.t. $W^{(1)} \in C(R_+;\mathcal{S}'_q)$ for $q > p + 3/2$:

Clearly

$$Q^{(1)}(\phi,\psi) = \langle D^2\phi, D^2\psi\rangle_p \quad \phi,\psi \in \Phi \tag{2.16}$$

then from Example 1.1 for $\phi \in \Phi$

$$Q^{(1)}(\phi,\phi) = \langle D^2\phi, D^2\phi\rangle_p = \sum_{n=1}^{\infty} (n + 1/2)^{2p}\langle D^2\phi,\phi_n\rangle_0^2$$

$$= \sum_{n=1}^{\infty} (n + 1/2)^{2p}\langle \phi, D^2\phi_n\rangle_0^2 \tag{2.17}$$

But from (1.1) with the notation of Example (1.1)

$$\frac{d}{dx}\phi_{n+1} = \sqrt{n!}\ \{\sqrt{g(x)}\ H_{n-1}(x) + H_n(x)[-\tfrac{x}{2}\, e^{-x^2/4}(2\pi)^{-1/4}]$$

$$= \sqrt{n!}\ \{\sqrt{g(x)}\ H_{n-1}(x) - \tfrac{x}{2}\sqrt{g(x)}\ H_n(x)\}$$

$$= \sqrt{n!}\ \sqrt{g(x)}\ H_{n-1}(x) + \frac{\sqrt{n!}}{2}\sqrt{g(x)}\{-(n+1)H_{n+1}(x) - H_{n-1}(x)\}$$

$$= \frac{\sqrt{n!}}{2}\frac{\phi_n}{\sqrt{(n-1)!}} - \frac{\sqrt{n!}}{2}(n+1)\frac{\phi_{n+2}}{\sqrt{(n+1)!}} = \frac{\sqrt{n}}{2}\phi_n - \frac{\sqrt{n+1}}{2}\phi_{n+2}$$

i.e.

$$\frac{d}{dx}\phi_{n+1} = \frac{\sqrt{n}}{2}\phi_n - \frac{\sqrt{n+1}}{2}\phi_{n+2} .$$

Hence

$$\frac{d^2}{dx^2}\phi_{n+1} = \frac{\sqrt{n(n-1)}}{4}\phi_{n-1} - \left(\frac{2n+1}{4}\right)\phi_{n+1} + \frac{\sqrt{(n+1)(n+2)}}{4}\phi_{n+3}$$

and

$$\langle\phi,D^2\phi_n\rangle = \frac{\sqrt{(n-1)(n-2)}}{4}\langle\phi,\phi_{n-2}\rangle_0 - \left(\frac{2n-1}{4}\right)\langle\phi,\phi_n\rangle_0 + \frac{\sqrt{n(n+1)}}{4}\langle\phi,\phi_{n+2}\rangle_0 \quad (2.19)$$

$$= a_n\langle\phi,\phi_{n-2}\rangle_0 + b_n\langle\phi,\phi_n\rangle_0 + c_n\langle\phi,\phi_{n+2}\rangle_0$$

where $a_n,b_n,c_n = O(n)$.

Then from this and (2.17)

$$Q^{(1)}(\phi,\phi) \le \alpha_1 \sum_n \left(n+\frac{1}{2}\right)^{2p+2}\langle\phi,\phi_{n-2}\rangle_0^2 + \alpha_2 \sum_n \left(n+\frac{1}{2}\right)^{2p+2}\langle\phi,\phi_n\rangle_0^2$$

$$+ \alpha_3 \sum_n \left(n+\frac{1}{2}\right)^{2p+2}\langle\phi,\phi_{n+2}\rangle_0^2$$

$$\le \alpha\|\phi\|_{p+1}^2, \quad (\alpha,\alpha_1,\alpha_2,\alpha_3 \text{ constants}).$$

Since the injection $\mathcal{S}_q \hookrightarrow \mathcal{S}_{p+1}$ is Hilbet-Schmidt for $q > p + 1 + 1/2 = p + 3/2$, we have shown that the Φ'-valued Wiener process given by (2.15) is such that

$$W^{(1)} \in C(R_+;\mathcal{S}_q') \text{ a.s. for } q > p + \frac{3}{2} .$$

Cylindrical Brownian motion

Definition 2.3 Let H be any real separable Hilbert space with norm $\|\cdot\|_H$. A family $\{B_t(h): t \in R_+, h \in H\}$ of real valued random variables is called a

<u>Cylindrical</u> <u>Brownian</u> <u>motion</u> (c.B.m.) on H if the following conditions hold:

(1) For each $h \in H$ $h \neq 0$, $\|h\|_H^{-1} B_t(h)$ is a one dimensional standard Wiener process.

(2) For any α_1, α_2 R and $h_1, h_2 \in H$

$$B_t(\alpha_1 h_1 + \alpha_2 h_2) = \alpha_1 B_t(h_1) + \alpha_2 B_t(h_2) \quad \text{a.s.}$$

(3) For each $t > 0$ and $h \in H$ $B_t(h)$ is an $\mathfrak{F}_t$-martingale where

$$\mathfrak{F}_t = \sigma(B_s(k): \ s < t \ \ k \in H) \ .$$

From (1) we have that $\forall t > 0$ and $h \in H$

$$E(B_t(h))^2 = t\|h\|_H^2$$

and therefore since by Sazonov's theorem $\exp(-\frac{t}{2}\|h\|^2)$ is not the characteristic functional of a countably additive probabiilty measure on H, there does not exist a process $\overline{B}_t$ in H s.t.

$$B_t(h) = \langle \overline{B}_t, h\rangle_H .$$

From (1), (2) and (3) we have that

$$E(B_t(h)B_s(k)) = \min(s,t) \ \langle h,k\rangle_H \ . \tag{2.18}$$

If $\{e_n\}_{n\geq 1}$ is a CONS in H, $B_t^n = B_t(e_n)$ is a sequence of independent one dimensional Brownian motions, and if $h \in H$

$$h = \sum_{n=1}^{\infty} \langle h,e_n\rangle_H \, e_n$$

for

$$h^{(n)} = \sum_{i=1}^{n} \langle h,e_i\rangle_H e_i$$

$$B_t(h^{(n)}) = \sum_{j=1}^{n} \langle h,e_j\rangle_H B_t^j.$$

For $n > m$,

$$E[B_t(h^{(n)}) - B_t(h^{(m)})]^2 = t \sum_{j=m+1}^{n} \langle h,e_j\rangle_H^2 \to 0 \quad m \to \infty$$

and therefore

$$B_t(h) = \sum_{j=1}^{\infty} \langle h,e_j\rangle_H B_t^j \quad \text{a.s.} \quad \forall t \geq 0 \text{ and } h \in H. \tag{2.19}$$

Conversely if $B_t^j (j \geq 1)$ is a sequence of independent Brownian motions then B_t defined by (2.19) is a c.B.m. on H. (Note that the RHS of (2.19) converges a.s. for each $t \geq 0$ and every $h \in H$).

The following result relates Φ'-valued Wiener processes and cylindrical Brownian motions.

Theorem 2.3 Let $W = (W_t)_{t \geq 0}$ be a Φ'-valued Wiener process with covariance functional Q. Then W defines a Rigged Hilbert space

$$\Phi \hookrightarrow H_Q \hookrightarrow \Phi'$$

and a cylindrical Brownian motion $\hat{W}$ on H_Q where H_Q is the Q-completion of Φ.

Proof. Since by assumption Q is a positive definite continuous bilinear symmetric form on $\Phi \times \Phi$, $Q(\cdot,\cdot)$ defines an inner product on $\Phi \times \Phi$. Let H_Q be the Q-completion of Φ. Since Φ is separable then H_Q is separable.

Let $\{\phi_j\}_{j \geq 1} \subseteq \Phi$ be a CONS for H_Q. Then $(W_t^j)_{j \geq 1}$, where $W_t^j = W_t[\phi_j]$, is a sequence of independent standard Wiener processes since

$$E(W_t^j W_s^k) = \min(s,t)Q(\phi_j,\phi_k) = \min(s,t)\delta_{jk}$$

For $h \in H_Q$ $h = \sum_{j=1}^{\infty} \langle h,\phi_j\rangle_Q \phi_j$,

$$\hat{W}_t[h] := \sum_{j=1}^{\infty} \langle h,\phi_j\rangle_Q W_t^j$$

defines a c.B.m. on H_Q as in (2.19).

Φ'-VALUED MARTINGALES

Most of the material in this section is taken from Mitoma [10].

Definition 2.4 A Φ'-valued process $M = (M_t)_{t \geq 0}$ is a Φ'-martingale w.r.t $(\mathcal{F}_t)$ if for each $\phi \in \Phi$ $M_t[\phi]$ is a martingale w.r.t.$(\mathcal{F}_t)$.

Since it will help in our later work, we shall also assume the additional condition

$$E(M_t)[\phi])^2 < \infty \quad \forall\phi \in \Phi \quad t > 0. \tag{2.20}$$

Theorem 2.4 Let M be a Φ'-valued martingale w.r.t. $(\mathcal{F}_t)$. Then there exists a Φ'-valued version $\tilde{M}$ of M s.t. the following conditions hold:

(1) For each $T > 0$ there exists $p = p_T > 0$ s.t.

$$\tilde{M}^T \in D([0,T];\Phi_p') \quad \text{a.s.}$$

where $D([0,T];\Phi_p')$ is the Skorohod space of right continuous functions with left hand limits (r.c.l.l.) from $[0,T]$ to Φ_p'.

(2) $\tilde{M}$ is r.c.l.l. in the strong Φ'-topology, i.e.

$$\tilde{M} \in D([0,\infty):\Phi') \quad \text{a.s.}$$

Proof. (1) Fix $T > 0$ and define $V_T^2(\phi) = E(M_T[\phi]^2)$. Then by Corollary 2.2 there exist $\theta = \theta_T > 0$ and $r = r_T > 0$ s.t.

$$V_T(\phi) \leq \theta \|\phi\|_r \quad \forall \phi \in \Phi \tag{2.21}$$

Let D be a countable dense subset of $[0,T]$. Then by (1.3) VI in [3], for $\phi \in \Phi$

$$E(\sup_{t \in D} M_t[\phi]^2) \leq 4 \sup_{0 \leq t \leq T} (E\, M_t[\phi]^2) = 4(E\, M_T[\phi]^2) . \tag{2.22}$$

Let $q > r$ be such that the injection map $\Phi_q \hookrightarrow \Phi_r$ is Hilbert-Schmidt, i.e. if $\{\phi_j\}_{j \geq 1} \subseteq \Phi$ is a CONS for Φ_q then

$$\sum_{j=1}^{\infty} \|\phi_j\|_r^2 < \infty.$$

Then from (2.21) and (2.22) we have

$$E\Big(\sum_{j=1}^{\infty} \sup_{t \in D} (M_t[\phi_j])^2\Big) = \sum_{j=1}^{\infty} E\big(\sup_{t \in D} (M_t[\phi_j])^2\big) \leq 4\theta^2 \sum_{j=1}^{\infty} \|\phi_j\|_r^2 < \infty.$$

So, if $\Omega_1 = \{\omega \in \Omega: \sum_{j=1}^{\infty} \sup_{t \in D} M_t[\phi_j]^2(\omega) < \infty\}$, $P(\Omega_1) = 1$.

Since each real valued martingale $M_t[\phi_j]$ has a right continuous modification X_t^j, writing

$$\Omega_t^j := \{\omega \in \Omega: X_t^j(\omega) = M_t[\phi_j](\omega)\}, \text{ we have}$$

$P(\Omega_t^j) = 1$ for $t \in D$. Then the set defined by

$$\Omega_2 := \Big(\bigcap_{t \in D} \bigcap_{j \geq 1} \Omega_t^j\Big) \cap \Omega_1$$

has probability one and if $\omega \in \Omega_2$

$$\sum_{j=1}^{\infty} \sup_{0 \leq t \leq T} (X_t^j(\omega))^2 < \infty.$$

Let $\{\tilde{\phi}_j\}_{j\geq 1}$ be the CONS of Φ'_q dual to $\{\phi_j\}_{j\geq 1}$ and for $0 \leq t \leq T$ define

$$\tilde{M}_t(w) = \begin{cases} \sum_{j=1}^{\infty} X_t^j(w)\tilde{\phi}_j & \omega \in \Omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Then for $0 \leq t \leq T$ $P(\tilde{M}_t \in \Phi'_q) = 1$ and $\tilde{M}_t(\omega)[\phi] = M_t(\omega)[\phi]$ for all $\phi \in \Phi$ $\omega \in \Omega_2$, i.e. $\tilde{M}_t = M_t$ a.s.

Next since for $s,t \in [0,T]$ and $j \geq 1$

$$|X_t^j(\omega) - X_s^j(\omega)|^2 \leq 4 \sup_{0\leq t\leq T} (X_t^j(\omega))^2, \quad (\omega \in \Omega_2),$$

by the dominated convergence theorem, fixing ω in Ω_2,

$$\lim_{s\downarrow t} \|\tilde{M}_t(\omega) - \tilde{M}_s(\omega)\|_{-q}^2 = \lim_{s\downarrow t} \|\sum_{j=1}^{\infty} (X_t^j(\omega) - X_s^j(\omega))\tilde{\phi}_j\|_{-q}^2$$

$$= \lim_{s\downarrow t} \sum_{j=1}^{\infty} |X_t^j(\omega) - X_s^j(\omega)|^2 = \sum_{j=1}^{\infty} \lim_{s\downarrow t} |X_t^j(\omega) - X_s^j(\omega)|^2 = 0,$$

the last assertion following from the right continuity of $X_t^j(\omega)$. In a similar fashion the fact that $\tilde{M}_t$ has left hand limits in the $\|\cdot\|_{-q}$-norm is shown.

Thus we have proved that for each $T > 0$ there exists $q_T > 0$ s.t. M_t has a r.c.l.l. version $\tilde{M}_t$ in the $\|\cdot\|_{q_T}$-norm, i.e.

$$\tilde{M}^T_{\cdot} \in D([0,T]; \Phi'_q).$$

(2) Let $T_n \uparrow \infty$, then by (1) there exists q_n s.t. M_t has a version $\tilde{M}^{T_n}$ with

$$\tilde{M}^{T_n}_{\cdot} \in D([0,T_n]; \Phi'_{q_n}) \quad \text{a.s.}$$

With the notation used in the proof of (1) let $\Omega_3 = \bigcap_{n=1}^{\infty} \Omega_2^n$. If $\omega \in \Omega_3$ define for $0 \leq t < \infty$

$$\tilde{M}_t(\omega) = \tilde{M}_t^{T_n}(\omega) \quad \text{for } T_{n-1} \leq t \leq T_n, \quad (T_0 = 0).$$

Then $P(\tilde{M} \in \Phi') = 1$ and $\tilde{M}_t(w) = M_t(w)$ for $\omega \in \Omega_3$.

Hence for $t > 0$ and $\varepsilon > 0$ there exists $\delta_t > 0$ s.t. if $t > s + \delta_t$

$$\|\tilde{M}_t(\omega) - \tilde{M}_s(\omega)\|_{-q_n} < \varepsilon$$

for $t < T_n$ and therefore for any bounded set $B \quad \Phi$

$$\sup_{\phi \in B} |(\tilde{M}_t(\omega) - \tilde{M}_s(\omega))[\phi]| < \varepsilon, \quad t > s + \delta_t$$

i.e. $\tilde{M}_t$ is strongly right continuous. A similar argument shows that it has left hand limits. Q.E.D.

Remarks

2.3) The above theorem can be proved without the assumption (2.20). The proof is very similar to the one given above using (1.1)VI of [3] instead of (2.22).

2.4) If M_t is a Φ'-valued martingale s.t. for each $\phi \in \Phi$

$$\sup_{0 \leq t < \infty} E(M_t[\phi]^2) < \infty, \tag{2.23}$$

there exists $q > 0$ s.t. M_t has a version $\tilde{M}_t \in D([0,\infty),\Phi_q')$ a.s. This is seen using the fact that if D is a countably dense subset of R_+ then from (2.22)

$$E(\sup_{t \in D} (M_t[\phi]^2) \leq 4E(M_\infty[\phi]^2) < \infty.$$

The next theorem is proved in a very similar way to Theorem 2.4.

Theorem 2.5 Let M be a Φ'-valued martingale such that for each $\phi \in \Phi$ $M_t[\phi]$ has a continuous version. Then there exists a Φ'-valued version $\tilde{M}$ of M such that the following conditions hold:

(1) For each $T > 0$ there exists $p = p_T > 0$ s.t.

$$\tilde{M}^T_{\cdot} \in C([0,T];\Phi'_p) \quad \text{a.s.}$$

(2) $\tilde{M}$ is continuous in the strong Φ'-topology i.e.

$$\tilde{M}_{\cdot} \in C([0,\infty);\Phi') \quad \text{a.s.}$$

(3) If $\sup\limits_{0 \leq t < \infty} E(M_t[\phi]^2) < \infty$ then there exists $p > 0$ s.t.

$$\tilde{M}_{\cdot} \in C([0,\infty): \Phi'_p) \quad \text{a.s.}$$

An example of a Φ'-martingale is the Φ'-valued Wiener process with

$$\mathcal{F}_t = \sigma(W_s[\phi];\ 0 \leq s \leq t,\ \phi \in \Phi)$$

and for which (Theorem 2.1) there exists a continuous version on $C([0,\infty);\Phi'_q)$ for some $q > 0$. This shows that condition (2.23) in Theorem 2.5(3) is not necessary for a Φ'-valued martingale to have a version in $C([0,\infty):\Phi'_p)$ for some $p > 0$.

The following example (due to G. Kallianpur and S. Ramaswamy) gives a Φ'-valued strongly continuous gaussian martingale M_t for which the following is not true: There exists p independent of t s.t.

$$M_t \in \Phi'_p \quad \forall t > 0, \quad \text{a.s.}$$

EXAMPLE 2.3 Consider the CHNS of Example 1.2 i.e. (Φ,H,L) is a special compatible family where $(I + L)^{-r_1}$ is a Hilbert-Schmidt operator for some $r_1 > 0$

$$L\phi_j = \lambda_j \phi_j$$

$$\sum_{j=1}^{\infty} (1 + \lambda_j)^{-2r_1} < \infty$$

for $\{\phi_j\}_{j\geq 1} \subseteq \Phi$ a CONS for H.

Define for $s \in [0,\infty)$ and $\phi \in \Phi$

$$f(s,\phi) = \sum_{j=1}^{\infty} (1 + \lambda_j)^s \langle\phi_j,\phi\rangle_0$$

Let $(B_s)_{s\geq 0}$ be a real valued standard Brownian motion. Since for each $t > 0$ and $\phi \in \Phi$

$$\int_0^t f(s,\phi)^2 ds < \infty$$

then the Wiener integral

$$X_{t,\phi} = \int_0^t f(s,\phi)dB_s$$

is a Gaussain martingale for each $\phi \in \Phi$. Since $f(s,\phi)$ is linear and continuous in Φ then the linear random functional

$$X_{t,\phi}: \Phi \to L_2(\Omega)$$

is Φ-continuous. Hence, by the regularization lemma there exists a Φ'-valued random variable X_t s.t.

$$X_t[\phi] = X_{t,\phi} \quad \text{a.s.} \quad \forall\phi \in \Phi$$

Then for $\mathcal{F}_t = \mathcal{F}_t^B$ $(X_t, \mathcal{F}_t)_{t\geq 0}$ is a Φ'-valued martingale such that $\forall\phi \in \Phi$ $X_t[\phi]$ has a continuous version. Hence by Theorem 2.5 X has a strongly continuous version also denoted by X.

Now suppose there exists $p > 0$ s.t. $X_t \in \Phi_p'$ a.s. $\forall t > 0$. Take $p > r_1$ and consider

$$\phi^{(n)} = \sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1} \phi_j \quad .$$

Then $\{\phi^{(n)}\}$ converges in Φ_p to ϕ say, and therefore $X_t[\phi^{(n)}] \underset{n\to\infty}{\to} X_t[\phi]$. But since X_t is L^2-continuous

$$E(X_t[\phi^{(n)}]^2) \underset{n\to\infty}{\to} E(X_t[\phi])^2 < \infty \quad \forall t > 0 \tag{2.24}$$

the finiteness of the limit being a consequence of $X_t[\phi]$ being a Gaussian random variable. On the other hand, if $t > p + r_1$,

$$E(X_t[\phi^{(n)}]^2) = E(\sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1} X_{t,\phi_j})$$

$$= \int_0^t (f(s, \sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1} \phi_j))^2 ds =$$

$$\int_0^t (\sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1+s})^2 ds$$

$$> \int_{p+r_1}^t (\sum_{j=1}^{n} (1 + \lambda_j)^{-p-r_1+s})^2 ds$$

Then by Fatou's lemma

$$\underline{\lim}\, E(X_t[\phi^{(n)}]^2) = \infty$$

which, in view of (2.24) implies that $E(X_t[\phi])^2 = \infty$, a contradiction.

STOCHASTIC INTEGRALS TAKING VALUES IN A DUAL OF A NUCLEAR SPACE

Consider a Φ'-valued right continuous martingale $(M_t)_{t\geq 0}$ w.r.t. a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that

$$E(M_t[\phi])^2 < \infty \quad \forall \phi \in \Phi \text{ and } t \geq 0.$$

Then for each $\phi \in \Phi$ there exists a predictable right continuous, non negative increasing process A_t^ϕ such that $X_t[\phi]^2 - A_t^\phi$ is a martingale, which implies that

$$E(X_t[\phi])^2 = EA_t^\phi \tag{2.25}$$

Let $\mathcal{L}^2(M)$ be the class of real valued predictable processes $f: R_+ \times \Omega \to R$ such that

$$E\int_0^t f_s^2(\omega)dA_s^\phi(\omega) < \infty \quad \forall \phi \in \Phi.$$

Before introducing the definition of the stochastic integral of a real valued process w.r.t. the Φ'-valued martingale M_t we prove the following lemma.

Lemma 2.3 For $f \in \mathcal{L}^2(M)$ and $t \geq 0$ define

$$V_t^f(\phi) = E\int_0^t f_s^2(\omega)dA_s^\phi(\omega) \tag{2.26}$$

Then V_t^f is a continuous function of Φ.

Proof. We first assume that f is a bounded function, i.e. $|f_t(\omega)| \leq K$ $\forall$ $t \geq 0$ and $\omega \in \Omega$. Then by (2.25) and Corollary 2.2

$$V_t^f(\phi) \leq K\, EA_t^\phi = KE(X_t[\phi])^2 \leq K\theta_t \|\phi\|_{r_t} \quad \forall \phi \in \Phi$$

which implies that $V_t^f(\phi)$ is Φ-continuous.

Next if $f \in \mathcal{L}^2(M)$ define

$$f_K(t,\omega) = f(t,\omega)1_{[-K,K]}(f(t,w)) \; t \geq 0, \quad \omega \in \Omega.$$

Then $|f_K(t,\omega)| \leq K$ and $f_K^2(t,\omega) \uparrow f^2(t,\omega) \; \forall \; t \geq 0, \omega \in \Omega$.

Denote by ν_ϕ the measure on the σ-field of predictable sets defined by the relation

$$\int g d\nu_\phi = E \int_0^t g_s dA_s^\phi$$

where g is a non-negative predictable function. We have, using the monotone convergence theorem that

$$E \int_0^t f^2 \, dA^\phi = \int_{[0,t]\times\Omega} f^2 d\nu_\phi = \lim_{k\to\infty} \int_{[0,t]\times\Omega} f_k^2 \, d\nu_\phi = \lim_{k\to\infty} E \int_0^t f_k^2 \, dA^\phi.$$

Hence $V_t^f(\phi)$ is the limit of the increasing sequence of continuous functions $V_t^{f_k}(\phi)$, and therefore $V_t^f(\phi)$ is lower semicontinuous.

Then the Baire category argument (Lemma 2.1) implies that $V_t^f(\phi)$ is Φ-continuous. Q.E.D.

We now define a Φ'-valued stochastic integral for $f \in \mathcal{L}^2(M)$.

<u>Definition 2.5</u> Let $f \in \mathcal{L}^2(M)$. For $\phi \in \Phi$ define

$$Y_t(\phi) := \int_0^t f_s \, dM_s[\phi]$$

where the RHS is the real valued stochastic integral w.r.t. the martingale $M_t[\phi]$. Then $Y_t(\phi)$ is a real valued martingale with a right continuous version and by Lemma 2.3

$$E(Y_t(\phi))^2 = E\int_0^t f_s^2 \, dA_s^\phi$$

is a continuous function of Φ. Hence the linear random functional $Y_t(\cdot): \Phi \to L^2(\Omega, \mathcal{F}, P)$ is continuous and by the regularization Lemma there exists $I_t(f) \in \Phi'$ a.s. such that

$$I_t(f)[\phi] = Y_t(\phi) \quad \text{a.s.} \quad \forall\ \phi \in \Phi, \quad t > 0.$$

By Theorem 2.4 there exists a version, also denoted by $I_t(f)$, which is r.c.l.l. in the strong Φ'-topology. The right continuous Φ'-valued martingale $I_t(f)$

$$I_t(f) := \int_0^t f_s \, dM_s$$

is defined to be the <u>stochastic integral of f w.r.t. the Φ'-valued martingale M_t</u>.

We now introduce a Φ'-valued stochastic integral of a Φ'-valued process w.r.t. a real valued martingale.

<u>Definition 2.6</u> Let $m = (m_t)$ be a real valued right continuous martingale such that $E(m_t^2) < \infty$ $\forall$ $t > 0$. Let $A_s = \langle m \rangle_s$ the integrable increasing process of M_t. Let $(F_t)_{t>0}$ be a predictable Φ'-valued process such that

$$E\int_0^t F_s[\phi]^2 dA_s < \infty \quad \forall \quad \phi \in \Phi \quad \text{and} \quad t > 0.$$

Write

$$Y_t(\phi) = \int_0^t F_s[\phi] dM_s$$

then

$$E(Y_t(\phi))^2 = E\int_0^t F_s[\phi]^2 \, dA_s$$

and by the Baire category argument, the regularization lemma and Theorem 2.4 there exists a Φ'-valued right continuous martingale $J_t(f)$ s.t.

$$J_t(f)[\phi] = Y_t(\phi) \quad \text{a.s.} \quad \forall\ \phi \in \Phi, \quad t \geq 0.$$

We define $J_t(F) := \int_0^t F_s dM_s$ as the Φ'-valued stochastic integral.

LECTURE III

ORNSTEIN-UHLENBECK STOCHASTIC DIFFERENTIAL EQUATIONS ON DUALS OF NUCLEAR SPACES

We now introduce a special class of linear stochastic differential equations with values in duals of nuclear spaces, namely Ornstein-Uhlenbeck type processes with a nuclear valued martingale as a driving term.

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a right continuous filtration $(\mathfrak{F}_t)_{t \geq 0}$. Let $\Phi \hookrightarrow H \hookrightarrow \Phi'$ be a rigged Hilbert space, $A: \Phi \to \Phi$ a continuous linear operator and $(T_t)_{t \geq 0}$ a strongly continuous semigroup such that (Φ, H, T_t, A) is a compatible family (see Lecture I). Let $(M_t)_{t \geq 0}$ be a Φ'-valued martingale (Definition 2.4) which is right continuous with left hand limits (r.c.l.l). Consider the stochastic differential equation

$$\begin{aligned} d\xi_t &= A'\xi_t dt + dM_t \quad t > 0 \\ \xi_0 &= \eta \end{aligned} \tag{3.1}$$

where η is a $\mathfrak{F}_0$-measurable Φ'-valued random variable and $A': \Phi' \to \Phi'$ is defined by the relation

$$A'f[\phi] = f[A\phi] \quad \forall \quad f \in \Phi', \quad \phi \in \Phi.$$

A special case of the SDE (3.1) is

$$\begin{aligned} d\xi_t &= A'\xi_t dt + dW_t \quad t > 0 \\ \xi_0 &= \eta \end{aligned} \tag{3.2}$$

where W_t is a Φ'-valued centered Wiener process (Definition 2.2) with covariance Q. From Theorem 2.1 $W_\cdot \in C(R_+ : \Phi'_q)$ for some $q > 0$ where q depends only on Q and not on t.

In this lecture we solve the SDE (3.2) (Theorem 3.1). The general martingale case (Theorem 3.2) was considered in Christensen [1].

Definition 3.1 We say that the SDE (3.2) has a Φ'-valued solution $\xi = (\xi_t)_{t \geq 0}$ if the following four conditions hold:

i) (ξ_t) is $\mathcal{F}_t$-adapted and Φ'-valued.

ii) $\xi \in C(R_+; \Phi')$ a.s.

iii) $\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + W_t[\phi] \quad \forall \ \phi \in \Phi$ a.s $\forall \ t \geq 0$.

iv) For each $T > 0$

$$E\left(\sup_{0 \leq t \leq T} |\xi_t[\phi]|^2\right) < \infty \quad \forall \ \phi \in \Phi$$

Remark 1. Condition (iv) above is implied by the following condition: for each $T > 0$

$$E \int_0^T (\xi_s[A_s\phi])^2 ds < \infty \quad \forall \ \phi \in \Phi$$

Proposition 3.1. If $(\xi_t)_{t \geq 0}$ is a solution of the SDE (3.2) then for each $T > 0$ there exist $q_T > 0$ and a version of ξ (denoted also by ξ) such that

$$\xi_\cdot^T \in C([0,T]; \Phi'_{q_T}) \quad \text{a.s.}$$

and

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + W_t[\phi] \quad \forall \ \phi \in \Phi, \ 0 \leq t \leq T \ \text{a.s.}$$

Proof. Given $T > 0$ define

$$G_T^2(\phi) := E\left(\sup_{0 \leq t \leq T} |\xi_t[\phi]|^2\right).$$

Then by condition (iv) in Defintion 3.1 $G_T(\phi) < \infty \ \forall \ \phi \in \Phi$ and clearly

$G_T(\phi_1 + \phi_2) < G_T(\phi_1) + G_T(\phi_2)$, $G_T(a\phi_1) = |a| G_T(\phi_1)$, $\phi_1, \phi_2 \in \Phi$, $a \in R$. Next $\sup_{0<t<T} |\xi_t[\phi]|$ is a lower semicontinuous function of Φ. Hence by Fatou's Lemma $G_T(\phi)$ is also a lower semicontinuous function of ϕ. Then by a Baire category argument there exist $\theta_T > 0$ and $r_T > 0$ s.t.

$$E(\sup_{0<t<T} |\xi_T[\phi]|^2) < \theta_T \|\phi\|^2_{r_T} \ \forall \ \phi \in \Phi$$

Let $p_T > r_T$ such that the injection map $\Phi_{p_T} \hookrightarrow \Phi_{r_T}$ is Hilbert-Schmidt and let $\{\phi_j\}_{j\geq 1} \subseteq \Phi$ be a CONS for Φ_{p_T} with dual basis $\{\hat{\phi}_j\}_{j\geq 1}$ a CONS for Φ'_{p_T}. Then

$$E(\sum_{j=1}^{\infty} \sup_{0<t<T} |\xi_t[\phi_j]|^2) < \theta_T \sum_{j=1}^{\infty} \|\phi_j\|^2_{r_T} < \infty$$

Define

$$\Omega_T = \{\omega: \sum_{j=1}^{\infty} \sup_{0<t<T} |\xi_t(\omega)[\phi_j]|^2 < \infty\}$$

Then $P(\Omega_T) = 1$. Next define

$$\hat{\xi}_t(\omega) = \begin{cases} \sum_{j=1}^{\infty} \xi_t(\omega)[\phi_j]\hat{\phi}_j & \omega \in \Omega_T \\ 0 & \omega \notin \Omega_T \end{cases}$$

Hence, $\hat{\xi}_t \in \Phi'_{p_T}$ $0 < t < T$ a.s. and $\hat{\xi}_t(\omega)[\phi] = \xi_t(\omega)[\phi]$ $\forall \phi \in \Phi$ and $\omega \in \Omega_T$. Moreover by the dominated convergence theorem if $t, t_0 \in [0,T]$

$$\lim_{t \to t_0} \|\hat{\xi}_t(\omega) - \hat{\xi}_{t_0}(\omega)\|^2_{-p_t} = \lim_{t \to t_0} \sum_{j=1}^{\infty} (\xi_t(\omega)[\phi_j] - \xi_{t_0}(\omega)[\phi_j])^2 = 0$$

i.e. $\hat{\xi}^T_{\cdot} \in C([0,T]; \Phi'_{p_T})$ a.s. and therefore

$$P(\omega: M_T(\omega) := \sup_{0<t<T} \|\hat{\xi}_t\|^2_{-p_T} < \infty) = 1.$$

From now on we will write ξ_t instead of $\hat{\xi}_t$.

Next for $\omega \in \Omega_T$ and $0 \leq t \leq T$ define

$$Y_t(\omega)[\phi] = \int_0^t \xi_s(\omega)[A_s\phi]ds$$

We will show that $Y^T(\omega) \in C([0,T];\Phi'_{\ell_T})$ for some $\ell_T > 0$:

$$|Y_t(\omega)[\phi]| \leq M_T(\omega) \int \|A_s\phi\|_{p_T}\, ds. \qquad (3.3)$$

Then using the continuity of the map $s \to A_s\phi \;\forall\; \phi \in \Phi$, by a Baire category argument there exist $\theta'_T > 0$ and $n_T > p_T$ s.t.

$$\sup_{0 \leq t \leq T} |Y_t(\omega)[\phi]|^2 \leq M_T^2(\omega)\; \theta' \|\phi\|^2_{n_T} \quad \forall\; \phi \in \Phi \;.$$

Then $Y_t(\omega) \in \Phi'_{n_T} \;\forall\; 0 \leq t \leq T \;\; \omega \in \Omega_T$. Let $\ell_T > n_T$ be such that $\Phi_{\ell_T} \hookrightarrow \Phi_{n_T}$ is Hilbert-Schmidt and let $\{e_j\}_{j\geq 1} \subseteq \Phi$ be a CONS for Φ_{ℓ_T} with dual basis $\{\hat{e}_j\}_{j\geq 1}$ a CONS for Φ'_{ℓ_T}. Then

$$\sum_{j=1}^{\infty} \sup_{0 \leq t \leq T} |Y_t(\omega)[e_j]|^2 \leq M_T^2(\omega)\theta_T^2 \sum_{j=1}^{\infty} \|e_j\|^2_{n_T} < \infty.$$

Hence since from (3.3) $Y_t(\omega)[\phi]$ is a continuous function of t on $0 \leq t \leq T$ for each $\phi \in \Phi$, by the dominated convergence theorem we have

$$\lim_{t \to t_0} \|Y_t(\omega) - Y_{t_0}(\omega)\|^2_{-\ell_T} = \lim_{t \to t_0} \sum_{j=1}^{\infty} (Y_t[e_j] - Y_{t_0}[e_j])^2 = 0 \quad t,t_0 \in [0,T]$$

i.e. $Y^T_{\cdot}(\omega) \in C([0,T];\; \Phi'_{\ell_T}) \quad \omega \in \Omega_T$.

Then we have shown that $\int_0^t A'_s\; \xi_s ds \in C([0,T];\Phi_{\ell_T})$ a.s. for some $\ell_T > 0$. Hence taking $q_T = \max(r_0, q, p_T, \ell_T)$ we have that

$$Z_t = \gamma + \int_0^t A'_s\; \xi_s ds + W_t \in C([0,T];\Phi'_{q_T}) \quad \text{a.s.}$$

Then by conditions (ii) and (iii) in Definition 3.1, for each $T > 0$

$$P(Z_t = \xi_t \quad 0 \leq t \leq T) = 1 \qquad \text{Q.E.D.}$$

Theorem 3.1 Let (Φ, H, T_t) be a compatible family and assume that there exists $r_0 > 0$ s.t.

$$E\|\eta\|^2_{-r_0} < \infty.$$

Then (3.2) has a unique solution given by

$$\xi_t := T_t'\eta + \int_0^t A' \; T_{t-s}W_s ds + W_t \quad \text{a.s.}$$

i.e.

$$\xi_t[\phi] = \eta[T_t\phi] + \int_0^t W_s[T_{t-s}A\phi]ds + W_t[\phi] \quad \forall \quad \phi \in \Phi \tag{3.4}$$

satisfying the following properties.

a) For each $T > 0$ there exists $p = p_T$ s.t.

$$\xi^T_{\cdot} \in C([0,T],\Phi'_{p_T}) \text{ a.s. and } E(\sup_{0 \leq t \leq T} \|\xi_t\|^2_{-p_T}) < \infty.$$

b) If $\{e_j\}_{j \geq 1} \subseteq \Phi$ is a CONS in Φ_p and $\xi^j_t := \xi_t[e_j]$ then

$\sum_{j=1}^{\infty} \xi^j_t e_j$ converges uniformly in $[0,T]$ in Φ'_p a.s.

c) If in addition (Φ,H,T) is a special compatible family (see Example 1.2 and Remark 1.1) from Corollary 2.3 we have that $q > r_1 + r_2$. Then the solution $\xi = (\xi_t)$ of the SDE (3.2) is such that $\xi \in C(R_+;\Phi'_p)$ a.s. where $p > \max(r_1 + r_2, r_0)$ is independent of t.

We will give the proof of (a) and (b). The proof of (c) is given in Kallianpur and Wolpert [7].

The following Lemma will be useful in the proof of the theorem.

Lemma 3.1 (Christensen [1]) For each $F \in \Phi'$ and $s < t$

$$F[T_{t-s}\phi] - F[\phi] = \int_s^t F[T_{u-s}A\phi]du = \int_s^t F[T_{t-u}A\phi]du \quad \forall\ \phi \in \Phi . \qquad (3.5)$$

Proof. Since we have a compatible family (Φ,H,T_t) $T_t|\Phi$ is a strongly continuous semigroup on the Φ-topology and if

$$\Psi = \{\phi \in \Phi\colon \lim_{\varepsilon\downarrow 0} \frac{T_t\phi - \phi}{\varepsilon} = A\phi \text{ in the } \Phi\text{-topology}\}$$

then Ψ is dense in Φ.

Then for all $\phi \in \Psi$ and $0 < s < t$,

$$\frac{d}{du} F[T_{u-s}\phi] = F[T_{u-s}A\phi] \text{ and}$$

$$F[T_{t-s}\phi] = F[\phi] + \int_s^t F[T_{u-s}A\phi]du$$

Next for some $q > 0$ $|F[T_{u-s}A\phi]| < \|F\|_{-q}\|T_{u-s}A\phi\|_q$. Define

$$G(\phi) = \sup_{s<u<t} \|T_{u-s}A\phi\|_q$$

Then by the continuity of T and A $G(\phi) < \infty$ $\forall$ $\phi \in \Phi$ and it is a lower semicontinuous function of Φ. Then by the Baire category argument there exist $\theta > 0$ and $r > 0$ s.t. $\forall$ $u \in [s,t]$

$$\|T_{u-s}A\phi\|_q < \theta\|\phi\|_r \quad \forall \quad \phi \in \Phi.$$

Hence, since F, T and A are continuous, by the dominated convergence theorem if $\phi_n \to \phi$ in Φ $\{\phi_n\} \subseteq \Psi$

$$\int_s^t F[T_{u-s}A\phi_n]du \underset{n\to\infty}{\to} \int_s^t F[T_{u-s}A\phi]du$$

The second equality in (3.5) follows in a similar way. Q.E.D.

Proof of Theorem 3.1 Let Ω_1 be such that $P(\Omega_1) = 1$ and $\omega \in \Omega_1 \Rightarrow W_\cdot(\omega) \in C(R_+;\Phi_q')$.

Step 1. For $t > 0$ the map $\phi \to \int_0^t W_s[T_{t-s}A\phi]ds$ is continuous and linear on Φ:

Let $\phi \in \Phi$, then

$$\left|\int_0^t W_s[T_{t-s}A\phi]ds\right| \leq \int_0^t \|W_s\|_{-q} \|T_{t-s}A\phi\|_q ds$$

$$\leq \sup_{0\leq s\leq t} \|W_s\|_{-q} \cdot R_t(\phi) \tag{3.6}$$

where

$$R_t(\phi) = \int_0^t \|T_{t-s}A\phi\|_q \, ds.$$

Next $\phi_n \to \phi$ in Φ implies $\|T_{t-s}A\phi_n\|_q \to \|T_{t-s}A\phi\|_q$. Then by Fatou's lemma

$$R_t(\phi) \leq \underline{\lim}\, R_t(\phi_n)$$

i.e. $R_t(\phi)$ is a lower semicontinuous function. For $a \in R$ clearly we have $R_t(a\phi) = |a|R_t(\phi)$ and for $\phi_1, \phi_2 \in \Phi$

$$R_t(\phi_1 + \phi_2) \leq R_t(\phi_1) + R_t(\phi_2) .$$

Hence since $R_t(\phi) < \infty \ \forall\ \phi \in \Phi$, by the Baire category argument $R_t(\phi)$ is continuous in Φ and there exist $\theta_t > 0$ and $r_t > 0$ s.t.

$$R_t(\phi) \leq \theta_t \|\phi\|_{r_t} \quad \forall\ \phi \in \Phi.$$

Then from (3.6), for each fixed $t > 0$

$$\left| \int_0^t W_s[T_{t-s}A\phi]ds \right| \leq \theta_t \sup_{0 \leq s \leq t} \|W_s\|_{-q} \cdot \|\phi\|_{r_t} \quad \forall \phi \in \Phi$$

and thus the map $\phi \to \int_0^t W_s[T_{t-s}A\phi]ds$ is continuous and linear on Φ. Write

$$\xi_t^1 = \int_0^t W_s[T_{t-s}A\phi]ds \ .$$

For each $t \geq 0$ define

$$\xi_t(\omega) = \begin{cases} T_t'\eta(\omega) + \xi_t^1(\omega) + W_t(\omega) & \text{if} \quad \omega \in \Omega_1 \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

<u>Step 2</u>. We check that ξ_t is a solution of (3.2). Let $\omega \in \Omega_1$.

Using Lemma 3.1 with $F = W_s$ and $\phi \to A\phi$ we have

$$W_s(\omega)[T_{t-s}A\phi] = W_s(\omega)[A\phi] + \int_s^t W_s(\omega)[T_{u-s}A^2\phi]du \tag{3.8}$$

Using again Lemma 3.2 with $F = \eta$ and $s = 0$ we obtain

$$\eta[T_t\phi] = \eta[\phi] + \int_0^t \eta[T_uA\phi]du \tag{3.9}$$

Substituting (3.8) and (3.9) in (3.4) we have

$$\xi_t(\omega)[\phi] = \eta(\omega)[\phi] + \int_0^t \eta(\omega)[T_uA\phi]du + W_t(\omega)[\phi]$$

$$+ \int_0^t W_s(\omega)[A\phi] + \int_0^t \int_s^t W_s(\omega)[T_{u-s}A^2\phi]duds$$

Interchaning order of integration in the last term of the last expression and using (3.4) we have

$$\xi_t(\omega)[\phi] = \eta(\omega)[\phi] + \int_0^t \eta(\omega)[T_u A\phi]du + W_t(\omega)[\phi]$$

$$+ \int_0^t W_u(\omega)[A\phi]du + \int_0^t \int_0^u W_s(\omega)[T_{u-s}A^2\phi]ds\ du$$

$$= \eta(\omega)[\phi] + \int_0^t \eta(\omega)[T_u A\phi]du + W_t(\omega)[\phi]$$

$$+ \int_0^t \{W_u(\omega)[A\phi] + \int_0^u W_s(\omega)[T_{u-s}A^2\phi]ds\}du$$

i.e. $\forall\ \phi \in \Phi$ and $0 \leq t \leq T$

$$\xi_t(\omega)[\phi] = \eta(\omega)[\phi] + \int_0^t \xi_u(\omega)[A\phi]du + W_t(\omega)[\phi] \quad \omega \in \Omega_1.$$

Hence condition (iii) in Definition 3.1 is satisfied. Observe that $(t,\omega) \to \xi_t(\omega)$ is $\mathcal{B}(\Phi')/\mathcal{B}(R_+) \times \mathcal{F}$-measurable and for each $t \geq 0$ ξ_t is $\mathcal{F}_t^{W,\eta}$-measurable where

$$\mathcal{F}_t^{W,\eta} = \sigma\{\eta[\phi], W_s[\phi]: s \leq t, \phi \in \Phi\} \vee \{P\text{-null sets}\}.$$

Next, from (3.4), the assumptions on W and η and Step 1 we have

$$E(\sup_{0\leq t\leq T} (\xi_t[\phi])^2) \leq 3E(\sup_{0\leq t\leq T} (\eta[T_t\phi])^2)$$

$$+ 3E(\sup_{0\leq t\leq T} (\int_0^t W_s[T_{t-s}A\phi]ds)^2)$$

$$+ 3E(\sup_{0\leq t\leq T} (W_t[\phi])^2$$

$$\leq c_T^2 \|\phi\|_{m_T}^2 < \infty \quad \forall\ \phi \in \Phi \qquad (3.10)$$

for some constants $c_T > 0$ and $m_T > 0$, which shows condition (iv) in Definition 3.1. In the next step we shall show that ξ_t satisfies (ii) in Definition 3.1.

Step 3 Let $\rho_T > m_T$ s.t. the map $\Phi_{\rho_T} \hookrightarrow \Phi_{m_T}$ is Hilbert-Schmidt and let $\{e_j\}_{j\geq 1} \subseteq \Phi$ be a CONS in Φ_{ρ_T} with dual basis $\{\hat{e}_j\}_{j\geq 1}$ a CONS in Φ'_{ρ_T}. Then using (3.10)

$$E\Big(\sum_{j=1}^{\infty} \sup_{0\leq t\leq T} (\xi_t[e_j])^2\Big) \leq C_T \sum_{j=1}^{\infty} \|e_j\|_{m_T}^2 < \infty.$$

Let $\Omega_2 = \{\omega: \sum_{j=1}^{\infty} \sup_{0\leq t\leq T} (\xi_t(\omega)[e_j])^2 < \infty\}$, then $P(\Omega_2) = 1$. Define

$$\tilde{\xi}_t(w) = \begin{cases} \sum_{j=1}^{\infty} \xi_t(\omega)[e_j]\hat{e}_j & \omega \in \Omega_2 \\ 0 & \omega \notin \Omega_2. \end{cases}$$

Then from (3.10)

$$E\Big(\sup_{0\leq t\leq T} \|\tilde{\xi}_t\|_{-\rho_T}^2\Big) = E\Big(\sup_{0\leq t\leq T} \sum_{j=1}^{\infty} (\xi_t[e_j])^2\Big) \leq C_T \sum_{j=1}^{\infty} \|e_j\|_{m_T}^2 < \infty$$

It remains to show that the series converges uniformly in $[0,T]$ in the Φ'_{ρ_T}-norm and that $\tilde{\xi}_t(\omega) = \xi_t(\omega)$ a.s. in $0 \leq t \leq T$. Define

$$S_n(t,\omega) = \sum_{j=1}^{n} \xi_t(\omega)[e_j]\hat{e}_j$$

then

$$\sup_{0\leq t\leq T} \|S_{n'}(t,\omega) - S_n(t,\omega)\|_{-\rho_T}^2 \to 0 \quad \text{as } n',n \to \infty$$

and $S_n(\cdot,\omega) \in C([0,T],\Phi_{-\rho_T})$. But since $C([0,T],\Phi_{-\rho_T})$ is a complete metric space there exists $\tau_t(\omega) \in C([0,T],\Phi_{-\rho_T})$ s.t.

$$\sup_{0\leq t\leq T} \|S_n(t,\omega) - \tau_t(\omega)\|_{-\rho_T} \to 0 \quad \text{as } n \to \infty$$

and

$$\|\mathcal{T}_t(\omega)\|^2_{-p_T} = \lim_n \|S_n(t,\omega)\|^2_{-p_T} = \lim_{n\to\infty} \sum_{j=1}^{n} (\xi_t(\omega)[e_j])^2$$

$$= \|\tilde{\xi}_t(\omega)\|^2_{-p_T} \quad .$$

Hence

$$E\|\tilde{\xi}_t(\omega)\|^2_{-p_T} = E\|\mathcal{T}_t\|^2_{-p_T}.$$

Next

$$E|\mathcal{T}_t[\phi] - S_n(t)[\phi]|^2 \leq E\|\mathcal{T}_t - S_n(t)\|^2_{-p_T} \|\phi\|^2_{p_T} \to 0$$

then

$$E\,\mathcal{T}_t[\phi] = \lim ES_n(t)[\phi] = \lim_{n\to\infty} \sum_{j=1}^{n} E\xi_t[e_j]\hat{e}_j[\phi]$$

$\therefore$

$$\mathcal{T}_t[\phi] = \lim_{n\to\infty} S_n(t)[\phi] = \sum_{j=1}^{\infty} \xi_t[e_j]\hat{e}_j[\phi] = \tilde{\xi}_t[\phi] \quad \text{a.s.}$$

Then we have shown that for each $T > 0$ $\exists$ p_T s.t. the following set has probability one

$$\Omega_T = \{\omega: \tilde{\xi}^T(\omega) \in C([0,T];\, \Phi'_{p_T})\}.$$

Then taking $T_n \uparrow \infty$ and $\overline{\Omega} = \bigcap_{n=1}^{\infty} \Omega_{T_n}$ we have that for $\omega \in \overline{\Omega}$

$$\tilde{\xi}(w) \in C(R_+;\Phi').$$

<u>Step 4</u> Uniqueness.

Suppose that there exists a Φ'-valued process $\overline{\xi} = (\overline{\xi}_t)$ that is also a solution of (3.2). Then by Proposition 3.1 for each $T > 0$ there exists a set Ω_3 of probability one such that if $\omega \in \Omega_3$, $\overline{\xi}^T(w) \in C([0,T];\Phi'_{n_T})$ for some $n_T > 0$ (WLOG take $p_T > n_T$) and

$$\overline{\xi}_t(\omega)[\phi] = \gamma(\omega)[\phi] + \int_0^t \overline{\xi}_s[A\phi]ds + W_t[\phi] \quad \forall \quad \phi \in \Phi, \quad 0 < t < T.$$

Fix $\omega \in \Omega_2 \cap \Omega_3$. Then, suppressing ω in the following, from the above expression we have that for $0 < s < t < T$ and $\phi \in \Phi$

$$W_s[T_{t-s}A\phi] = \overline{\xi}_s[T_{t-s}A\phi] - \eta[T_{t-s}A\phi] - \int_0^s \overline{\xi}_u[AT_{t-s}A\phi]du. \tag{3.11}$$

Substituting the last expression in the second term of RHS of (3.4) we obtain

$$\xi_t[\phi] = \eta[T_t\phi] + \int_0^t \overline{\xi}_s[T_{t-s}A\phi]ds - \int_0^t \eta[T_{t-s}A\phi]ds$$

$$- \int_0^t \int_0^s \overline{\xi}_u[AT_{t-s}A\phi]duds + W_t[\phi]$$

and using Fubini's theorem

$$\xi_t[\phi] = \eta[T_t\phi] + \int_0^t \overline{\xi}_s[T_{t-s}A\phi]ds - \int_0^t \eta[T_{t-s}A\phi]ds$$

$$- \int_0^t \int_u^t \overline{\xi}_u[AT_{t-s}A\phi]dsdu + W_t[\phi] \tag{3.12}$$

Next, applying Lemma 3.1 to $F = \gamma$ we have

$$\int_0^t \eta[T_{t-s}A\phi]ds = \eta[T_t\phi] - \eta[\phi]\ . \tag{3.13}$$

Next since $AT_{t-s}\phi = T_{t-s}A\phi \ \ \forall \ \ \phi \in \Phi$, applying Lemma 3.1 to $F = \overline{\xi}_u$ we have

$$\int_u^t \overline{\xi}_u[AT_{t-s}A\phi]ds = \int_u^t \overline{\xi}_u[T_{t-s}A^2\phi]ds = \overline{\xi}_u[T_{t-u}A\phi] - \overline{\xi}_u[A\phi] \tag{3.14}$$

Then using (3.14) and (3.13) in (3.12)

$$\xi_t[\phi] = \eta[\phi] + \int_0^t \overline{\xi}_s[T_{t-s}A\phi]ds - \int_0^t \overline{\xi}_u[T_{t-u}A\phi]du$$

$$+ \int_0^t \overline{\xi}_u[A\phi]du + W_t[\phi] = \eta[\phi] + \int_0^t \overline{\xi}_u[A\phi]du + W_t[\phi] = \overline{\xi}_u[\phi].$$

Thus for each $T > 0$

$$\xi_t(\omega)[\phi] = \overline{\xi}_t(\omega)[\phi] \quad \forall\phi \in \Phi, \quad 0 < t < T \quad \omega \in \Omega_2 \cap \Omega_3$$

Hence we have shown that for each $T > 0$ there exists a set Ω_T of probability one given by

$$\Omega_T = \{\omega: \xi_t(\omega) = \overline{\xi}_t(\omega) \quad 0 < t < T\}.$$

Let $T_n \uparrow \infty$ and define $\overline{\Omega} = \bigcap_{n=1}^{\infty} \Omega_{T_n}$. Then $P(\overline{\Omega}) = 1$ and if $\omega \in \overline{\Omega}$

$$\xi_t(\omega) = \overline{\xi}_t(\omega) \qquad \forall \quad t > 0$$

i.e.

$$P(\xi_t = \overline{\xi}_t \quad t > 0) = 1.$$

The proof of the Theorem is complete.

Q.E.D.

Remarks.

3.1 In the case of a special compatible family (Φ,H,T_t) (Example 1.2), for each $j > 1$ ξ_t^j is the one dimensional Ornstein-Uhlenbeck process satisfying

$$d\xi_t^j = -\lambda_j \xi_t^j dt + dW_t[\phi_j]$$

$$\xi_0^j = \eta_0[\phi_j].$$

3.2 In applications one usually deals with an SDE of the form

$$d\tau_t = (-L\tau_t + m)dt + dW_t = L\tau_t dt + dZ_t$$

where Z_t is a Φ'-valued Wiener process with parameters (m,Q), with $m \in \Phi'$, i.e.

$$EZ_t[\phi] = tm[\phi].$$

In this case we have

$$|m[\phi]|^2 + Q(\phi,\phi) \le \theta\|\phi\|_{r_2}^2 \quad \forall \ \phi \in \Phi$$

for some $\theta > 0$ and $r_2 > 0$. Hence in the case of a special compatible family

$$d\tau_t^j = (-\lambda_j\tau_t^j + m_j)dt + dW_t[\phi_j]$$

$$\tau_o^j = n[\phi_j]$$

where $m_j = m[\phi_j]$, $\tau_t^j = \tau_t[\phi_j]$.

The following result can be shown in a similar way to Theorem 3.1

Theorem 3.2. Let $M = (M_t)_{t\geq 0}$ be a right continuous Φ'-valued martingale w.r.t. $(\mathfrak{F}_t)$ such that

$$E(M_t[\phi])^2 < \infty \quad \forall \ t \text{ and } \forall \ \phi \in \Phi.$$

Then the SDE (3.1) has a unique solution given by

$$\xi_t = T_t'n + M_t + \int_0^t A'T_{t-s}' M_s ds \quad \text{a.s.}$$

i.e.

$$\xi_t[\phi] = \eta[T_t\phi] + M_t[\phi] + \int_0^t M_s[T_{t-s}A\phi]ds \tag{3.15}$$

with the following property:

a) For each $T > 0$ there exists $\Omega_T \in \mathcal{F}$ with $P(\Omega_T) = 1$ and $p_T > 0$ s.t.

$$\xi^T_\cdot(\omega) = \{\xi_t(\omega): 0 \le t \le T\} \in D([0,T];\Phi'_{p_T}) \quad \forall \quad \omega \in \Omega_T.$$

EXAMPLE 3.1 (Poisson driven OU-SDE).

Consider a special compatible family (Φ,H,T_t) as in Example 1.2 where $H = L^2(x,d\Gamma)$ for some $\mathfrak{X}$ and a σ-finite measure Γ. In neurophysiological applications (see [7]) $\mathfrak{X}$ represents the surface membrane of a neuron, e.g. $\mathfrak{X}$ is taken as in Example 1.3 to be $[0,b]$.

Define the Φ'-valued martingale

$$Y_t[\phi] = \int_0^t \int_{R\times\mathfrak{X}} a\phi(x)\tilde{N}(da\ dx\ ds) \tag{3.16}$$

where

i) $\tilde{N}(da\ d\eta\ ds) = N(da\ d\eta\ ds) - \mu(da\ d\eta)ds$

and

ii) N is a poisson random measure, i.e. $N([0,t] \times A \times B)$ is a Poisson random variable with parameter $t\cdot\mu(A\times B)$ for $A \in B(R)$ and B is a measurable subset of $\mathfrak{X}$.

The interpretation is that $N([0,t] \times A \times B)$ = the number of voltage pulses of size $a \in A \subseteq R$ arriving at sites $x \in B$ at times $s \le t$.

Then

$$E(Y_t[\phi]Y_t[\psi]) = \int_0^t \int_{R\times\mathfrak{X}} a^2\phi(x)\psi(x)\mu(da,dx)ds$$

$$= t\ Q(\phi,\psi)$$

where

$$Q(\phi,\phi) = \int_{R\times\mathfrak{X}} a^2(\phi(x))^2\mu(da\ dx) \tag{3.17}$$

In [7] the semigroup (T_t) represents the evolution semigroup describing the decay of the difference ξ_t between the actual voltage potential at the time $t > 0$ and the resting potential on $\mathfrak{X}$, and this difference is modeled as the Φ'-valued solution of the SDE

$$d\xi_t = -L'\xi_t dt + dY_t, \quad E\|\eta\|_{-r_0} < \infty \qquad \text{some } r_0 > 0 \tag{3.18}$$

where $A = -L$ is the generator of T_t.

In this situation it can be shown that there exists $p > 0$ s.t. for each $T > 0$

$$\xi^T_\cdot \in D([0,T];\Phi'_p) \quad \text{a.s.}$$

(see [7]) and that $\sum_j \xi^j_t\phi_j$ converges uniformly in $[0,T]$ to $\xi^T_\cdot$ in the Φ'_p topology where

$$\xi^j_t = e^{-t\lambda_j}\ \xi_0[\phi_j] + \int_0^t e^{-\lambda_j(t-s)} dY_s[\phi_j] \tag{3.19}$$

and λ_j, ϕ_j satisfy (as in Example 1.2) $L\phi_j = \lambda_j\phi_j$ $j \geq 1$.

LECTURE IV

WEAK CONVERGENCE OF SOLUTIONS

At the end of the last lecture (Example 3.1) it was shown how the membrane voltage potential at time t ξ_t of a neuron can be modeled as a Φ'-valued stochastic differential equation driven by a stochastic process with stationary independent increments defined through a Poisson random measure. However, it is believed that the pulse sizes are quite small, making it reasonable to hope that they can be modeled by a Gaussian noise process. Let us now consider the weak convergence of the solutions of (3.14) to the corresponding SDE driven by a Gaussian noise. Most of the material in this lecture is taken from [2] and [7]. Sufficient conditions for the weak convergence of Φ'-valued stochastic processes are given in [11].

Let (Φ, H, T_t) be a compatible family and $\Lambda \in \mathcal{B}(\Phi')$. For each $n \geq 1$ let μ^n be a measure on $(R \times \Lambda, \mathcal{B}(R) \times \mathcal{B}(\Lambda))$ such that the positive definite bilinear form

$$Q^n(\phi,\psi) := \int_{R\times\Lambda} a^2\, \eta[\phi]\eta[\psi]\mu^n(da,d\eta) \tag{4.1}$$

is continuous on $\Phi \times \Phi$, and let N^n be a Poisson random measure with intensity measure $\mu^n(da,d\eta)dt$. Define

$$\tilde{N}^n(da\, d\eta\, ds) := N^n(da\, d\eta\, ds) - \mu^n(da\, d\eta)ds \tag{4.2}$$

and

$$Y^n[\phi] := \int_0^t \int_{R\times\Lambda} a\eta[\phi]\tilde{N}^n(da\, d\eta\, ds) . \tag{4.3}$$

Y_t defined in Example 3.1 is a special case of the above for a large class of spaces $\mathcal{X}$.

For $n \geq 1$ let $m^n \in \Phi'$ and consider the Φ'-valued process $\xi^n_\cdot$ given by

$$d\xi^n_t = A' \xi^n_t dt + m^n dt + dY^n_t$$

$$\xi^n_0 = \eta^n \tag{4.4}$$

where η^n is $\mathfrak{F}_0$-measurable.

The following result is proved in [7].

<u>Theorem 4.1</u> Assume the following six conditions hold:

1) There exists $r_2 > 0$ and $c > 0$ such that for $n \geq 1$

$$(m^n[\phi]^2) + Q^n(\phi,\phi) \leq c\|\phi\|^2_{r_2} \quad \forall \phi \in \Phi.$$

2) $\lim_{n\to\infty} Q^n(\phi,\phi) = Q(\phi,\phi) \quad \forall \phi \in \Phi$ for some positive definite bilinear continuous form Q on $\Phi \times \Phi$.

3) $\lim_{n\to\infty} m^n[\phi] = m[\phi] \quad \forall \phi \in \Phi$ for some $m \in \Phi'$.

4) There exists $r_3 > 0$ such that

$$\sup_n \max\{E\|\eta^n\|^2_{-r_0}, E\|\eta\|_{-r_0}\} < \infty.$$

5) ξ^n_0 converges in law to η on Φ'_{r_3} for some $\eta \in \Phi'_{r_0}$.

6) $\lim_{n\to\infty} \int_{R\times\Lambda} |a\eta[\phi]|^3 \mu^n(da\, d\eta) = 0 \quad \forall \phi \in \Phi$.

Then for each $T > 0$ there exists $p_T > 0$ such that $\xi^{n,T}$ converge weakly to ξ^T on $D([0,T],\Phi_{p_T})$ where ξ is the unique solution of

$$d\xi_t = A'\xi_t dt + mdt + dW_t$$
$$\xi_0 = \eta \tag{4.5}$$

and W is a centered Φ'-valued Wiener process with covariance functional Q. Furthermore

$$\xi^T \in C([0,T];\ \Phi'_{p_T}). \tag{4.6}$$

We conclude this lecture by discussing several important examples occurring in applications. They are discussed in [7], [15] and [16].

<u>EXAMPLE 4.1</u> (<u>White noise current injection at a single point</u> x_0).

Let $H = L^2([0,b],dx), T_t, -L$ and Φ be as in Example 3.1, i.e.
$\mathscr{X} = [0,b], b = \pi$, $H = L^2(x, \Gamma = \text{Leb}), L\phi = -\alpha\phi + \beta\Delta\phi$,
$\phi_j(x) = c_j(\cos(jx), c_0 = 1/\sqrt{\pi}, c_j = \sqrt{2/\pi}$ and $\lambda_j = a + \beta(j)^2$.

Assume that the impulses can arrive only at a single point $x_0 \in [0,b]$ with arrival rate measures μ^n of the form

$$\mu^n(A \times B) = \mu_1^n(A) \cdot 1_B(x_0) \quad A \in \mathcal{B}(R_+),\ B \in \mathcal{B}((0,b]) \tag{4.7}$$

where

$$\mu_1^n(A) = \sum_{k=1}^{p} f_e^{k,n}\, 1_A(a^{k,n}) + \sum_{\ell=1}^{q} f_i^{\ell,n}\, 1_A(-a_i^{\ell,n}) \tag{4.8}$$

and $a_e^{k,n} \in (0,\infty)$ are the possible sizes of "excitatory" pulse (positive) and $-a_i^{\ell,n}$ are the sizes of "inhibitory" pulses (negative), and $f_e^{k,n}$, $f_i^{k,n}$ are intensities of Poisson processes.

Write:

$$\sigma^2 := \sum_{k=1}^{p} f_e^{k,n}\, (a_e^{k,n})^2 + \sum_{\ell=1}^{q} f_i^{\ell,n} (a_i^{\ell,n})^2 \tag{4.9}$$

$$\gamma_n := \sum_{k=1}^{p} f_e^{k,n} a_e^{k,n} - \sum_{\ell=1}^{q} f_i^{\ell,n} a_i^{\ell,n} \tag{4.10}$$

and

$$X_t^n[\phi] = \gamma^n \phi(x_o) + Y_t^n[\phi] \quad \phi \in \Phi \tag{4.11}$$

where

$$Y_t^n[\phi] = \sum_k a_e^{k,n} \int_0^t \int_{\mathcal{X}} \phi(x)\, \tilde{N}_e^{k,n}(dx,ds)$$

$$- \sum_{\ell} a_i^{\ell,n} \int_0^t \int_{\mathcal{X}} \phi(x)\, \tilde{N}_i^{\ell,n}(dx,ds) \tag{4.12}$$

and $\tilde{N}_e^{k,n}, \tilde{N}_i^{\ell,n}$ are independent Poisson random measures with variance measures given by $f_e^{k,n} \nu(dx)$, $f_i^{\ell,n} \nu(dx)$ with $\nu(B) = 1_B(x_o)$.

Hence from (4.1) we have that

$$Q^n(\phi,\psi) = \phi(x_o)\psi(x_o)\sigma_n^2 \tag{4.13}$$

It is worth to observe that $\{\phi_j(x)\}_{j \geq 1}$ does not diagonalize Q^n.

For each $n \geq 1$ consider the SDE

$$\begin{aligned} d\xi_t^n &= -L' \xi_t^n dt + dX_t^n \\ \xi_o^n &= \eta_o^n. \end{aligned} \tag{4.14}$$

An application of Theorem 4.1 gives the following weak convergence of solutions of (4.14).

<u>Proposition 4.1.</u> Assume the following four conditions hold:

1) $\lim_{n\to\infty} \max_{k,\ell} \{a_e^{k,n}, a_i^{\ell,n}\} = 0.$

2) $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$ for some $0 < \sigma^2 < \infty$.

3) $\lim_{n\to\infty} \gamma_n = \gamma$ for some $\gamma < \infty$.

4) η_0^n converges in law to η_0, where η_0 is a Φ'_{r_0}-valued gaussian random variable for some $r_0 > 0$

Then $\lim Q^n(\phi,\psi) = Q(\phi,\psi) := \phi(x_0)\psi(x_0)\sigma^2$ and ξ^n converges weakly to ξ where ξ is the unique solution of the SDE

$$\begin{aligned} d\xi_t &= -L'\xi_t dt + dW_t \\ \xi_0 &= \eta_0 \end{aligned} \tag{4.15}$$

where W is a centered Φ'-valued Wiener process with covariance Q. Moreover using Example 1.3 and Theorem 3.1(c) we have that $W_\cdot \in C(R_+;\Phi'_p)$ for $p > 1/4$ and $\xi \in C(R_+;\Phi'_q)$ where $q > \max(1/4, r_0)$.

Furthermore $\xi^j := \xi_t[\phi_j]$ satisfies the real valued Ornstein-Uhlenbeck SDE

$$\begin{aligned} d\xi_t^j &= [-\lambda_j \xi_t^j + \gamma\phi_j(x_0)]dt + \sigma|\phi_j(x_0)|dW_t^j \\ \xi_0^j &= \xi_0[\phi_j] \end{aligned} \tag{4.16}$$

where the one dimensional standard Wiener processeses $W_t^j = W_t[\phi_j]$ $j \geq 1$ are not independent.

EXAMPLE 4.2 (White noise current uniformly distributed over $\mathcal{X}$).

Consider the previous example but now assuming that white noise injection can occur at any point in $[0,b]$ with arrival rate measure

$$\mu^n(A \times B) = \mu_1^n(A)\nu^n(B) \quad A \in \mathcal{B}(R_+), \quad B \in \mathcal{B}([0,\pi]) \tag{4.17}$$

where μ_1^n is defined in (4.8) and ν^n is a sequence of probability measures on $\mathcal{X} = [0,b]$ such that ν^n converge weakly to ν where ν is 1/b Lebesque measure. Then for each $n > 1$

$$Q^n(\phi,\psi) = \sigma^2 \int_0^b \phi(x)\psi(x)\nu^n(dx) \qquad \phi,\psi \in \Phi \tag{4.18}$$

In this case the corresponding weak convergence result is given by the following proposition.

Proposition 4.2. Assume that the conditions (1)-(4) of Proposition 4.1 are satisfied. Then

a) $\lim_{n\to\infty} Q^n(\phi,\psi) = Q(\phi,\psi) := \frac{\sigma^2}{b}\int_0^b \phi(x)\psi(x)dx.$

b) ξ_n converge weakly to ξ.

where ξ_t is the unique solution of (4.15) satisfying the same conditions as in Proposition 4.1 with the difference that W is a Φ'-valued Wiener process with covariance functional Q given by (a).

The Wiener process W can, in this instance, be defined in terms of the centered 2-parameter Wiener process $W_{t,x}$ $(0 < t < T,\ 0 < x < b)$ with covariance

$$E(W_{t,x}W_{s,y}) = \frac{1}{b}\min(t,s)\min(x,y), \tag{4.19}$$

by

$$W_t[\phi] := \frac{1}{b}\int_0^b \phi(x)d_xW_{t,x}.$$

Further properties of the solution ξ_t in this case have been investigated in detail by J. Walsh [15].

Theorem 3.1(c) and Corollary 2.3 are examples of results for which in the case of a special compatible family one can obtain additional information on the spaces where some Φ'-valued processes lie. This information is given in terms of r_1 (which satisfies condition (1.3) in Example 1.2) and of r_2 given in Corollary 2.3 and related to the covariance functional Q. The following Lemma is useful in determining r_2 when Q is of the form presented in the previous examples.

<u>Lemma 4.1</u> Let (Φ,H,T_t) be a special compatible family. Suppose there exists $r > 0$ such that

$$c_1 := \sup_x \sup_j |\phi_j(x)|(1 + \lambda_j)^{-r} < \infty \quad \forall \quad \phi \in \Phi$$

and

$$c_2 := \int_{R\times \mathcal{X}} a^2 \mu(da,dx) < \infty$$

Then

$$Q(\phi,\psi) = \int_{R\times \mathcal{X}} a^2 \phi(x)\psi(x)\mu(da,dx) \tag{4.20}$$

satisfies

$$Q(\phi,\phi) < \theta\|\phi\|^2_{r+r_1} \tag{4.21}$$

where $(\lambda_j)_{j\geq 1}$ and r_1 are given in Example 1.2.

The proof of the Lemma is given in [7].

EXAMPLE 4.3

Let $\mathcal{X} = S^2$, the unit sphere in R^3, with Lebesgue surface measure Γ. Let Δ_B be the Laplace-Beltrami operator

$$\Delta_B \phi = (\sin\theta)^{-1}\left[\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\phi}{\partial\theta} + \frac{\partial}{\partial\eta}(\sin\theta)^{-1}\frac{\partial\phi}{\partial\eta}\right] \tag{4.22}$$

(where θ, η are the Euler angles on S^2). Let $L = -\beta + \delta\Delta_B$ for $\beta,\delta > 0$ and $H = L^2(\mathcal{X},\Gamma)$. This time the eigenfunctions are the spherical harmonics $Y_{\ell m}(\ell=0,1,\ldots;m = -\ell,\ldots,\ell)$ with eigenvalues $\lambda = \beta + \delta\ell(\ell + 1)$ for L, $e^{-t\lambda}$ for T_t.

Write

$$\phi_j = Y_{\ell m}, \quad j = m + \ell(\ell + 1) \tag{4.23}$$

(i.e. $\ell = [\sqrt{j}]$, $m = j - \ell^2 - \ell$)

$$\lambda_j = \beta + \delta[\sqrt{j}]([j] + 1) . \tag{4.24}$$

Then

$$\sum_j (1 + \lambda_j)^{-2r_1} = \sum_\ell (2\ell + 1)[1 + \beta + \delta(\ell^2 + \ell)]^{2r_1} < \infty$$

if $r_1 > 1/2$. Hence condition (1.3) in Example 1.2 is satisfied and a special compatible family (Φ,H,T_t) can be constructed. Furthermore

$$\sup_x |Y_{\ell m}(x)|^2 = \left(\frac{2\ell + 1}{4\pi}\right)$$

and therefore

$$\sup_j \sup_x |\phi_j(x)| \quad (1 + \lambda_j)^{-1/4} = [4\pi^2 \min(\delta,4\beta)]^{-1/4} < \infty .$$

Hence c_1 in Lemma 4.1 is finite for $r = 1/4$. Then by Lemma 4.1 for any measure μ on $R \times \mathcal{X}$ such that

$$\int_{R\times\mathcal{X}} a^2\mu(da,dx) < \infty \tag{4.25}$$

we have that if Q is given by (4.26)

$$Q(\phi,\phi) \leq \theta\|\phi\|^2_{r+r_1} \qquad \forall \quad \phi \in \Phi$$

where $r + r_1 > 3/4$, i.e. $r_2 = r + r_1$. Then if q is as in Corollary 2.3 $q > r_1 + r_2 = \frac{1}{2} + \frac{3}{4}$, i.e. $q > \frac{5}{4}$.

A more general result is the following:

Proposition 4.3 Let $\mathcal{X}$ be a smooth d-dimensional compact Riemannian manifold with smooth (possible empty) boundary $\partial\mathcal{X}$ and Riemannian volume element $d\Gamma$. Let L be a positive, self-adjoint operator on a domain $\mathcal{D} \subseteq H = L^2(\mathcal{X},d\Gamma)$ such that

a) $C_0^\infty(\mathcal{X}) \subseteq \mathcal{D}$ (if ∂x is not empty, $\mathcal{X} \cup \partial\mathcal{X}$ is compact and $C_0^\infty(\ ) = C^\infty$-functions on $\mathcal{X}$ whose support lies in $\mathcal{X}$. If $\partial\mathcal{X}$ is empty, is compact and $C_0^\infty(\mathcal{X}) = C^\infty(\mathcal{X})$).

b) $L_0 := L|_{C_0^\infty(\ )}$ is a uniformly strongly elliptic differential operator of order $2m > 0$ with smooth coefficients.

c) $\mathcal{D} \subseteq W_{2m}(\mathcal{X})$, the Hilbert space of those elements in H with $2m$ weak derivatives in H.

Then L admits a CONS $\{\phi_j\}$ of eigenfunctions in H with eigenvalues $\{\lambda_j\}$ satisfying

i) $L\phi_j = \lambda_j \phi_j$

ii) $\phi_j \in C^\infty(\mathcal{X})$

iii) $\sum_j (1 + \lambda_j)^{-2r_1} < \infty$ for $r_1 > \frac{d}{4m}$

iv) $\sup_j \sup_x |\phi_j(x)|(1 + \lambda_j)^{-r} < \infty$ all $r > \frac{d}{4m}$

Hence by condition (iii) above and (1.3) we have that a special compatible family can be constructed in the manner of Example (1.2). Moreover from (iii) above if μ is any measure on $R \times \mathcal{X}$ satisfying (4.25), by Lemma 4.1 $r_2 > d/2m$. Then if q is as in Corollary 2.3 $q > r_1 + r_2$ i.e. $q > 3d/4m$.

Additional examples and applications of Φ'-valued processes can be seen in [7] and references therein.

LECTURE V

STOCHASTIC EVOLUTION EQUATIONS AND NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

In this lecture we will give an outline of recent works on stochastic evolution equations and nonlinear stochstic differential equations on the dual of a Countably Hilbert nuclear space.

Throughout this lecture $(\Omega, \mathfrak{F}, P)$ will denote a complete probability space with a right continuous filtration $(\mathfrak{F}_t)_{t \geq 0}$ and $(\Phi, \|\cdot\|_r, r \geq 0)$ will be a Countably Hilbert nuclear space.

1. STOCHASTIC EVOLUTION EQUATIONS

The material of this section is recent joint work with V. Perez-Abreu.

There are several possible extensions of the SDE (3.1). For example one may consider an evolution operator A_t instead of the infinitesimal generator $-L$ and/or a perturbation operator P_t. In this section we consider the SDE

$$d\xi_t = (A_t' \xi_t + P_t' \xi_t)dt + dW_t \qquad \text{(I)}$$

$$\xi_0 = \gamma$$

where γ is a $\mathfrak{F}_0$-measurable Φ'-valued gaussian random variable s.t. $E\|\gamma\|^2_{-r_0} < \infty$ some $r_0 > 0$ and W_t is a Φ'-valued Wiener process with covariance Q. By Theorem 2.1. $W. \in C(R, \Phi_q')$ a.s. for some $q > 0$. The operators A_t and P_t from Φ to Φ are assumed to satisfy the following conditions:

<u>Assumptions on</u> A_t:

a) For each $t \geq 0$ $A_t: \Phi \to \Phi$ is a continuous linear operator.

b) For each $\phi \in \Phi$ the map $t \to A_t\phi$ is continuous.

c) A_t is the generator of a two parameter semigroup $T(s,t)$ $0 < s < t < \infty$ $(T(s,t) = T(s,t')T(t',t) \quad s < t' < t, \; T(t,t) = I)$, i.e.

$$\frac{d}{dt} T(s,t)\phi = T(s,t)A_t \phi \quad \forall \phi \in \Phi \quad 0 < s < t$$

and

$$\frac{d}{ds} T(s,t)\phi = -A_s T(s,t)\phi \quad \forall \phi \in \Phi \quad 0 < s < t .$$

d) For $s < t$ $T(s,t): \Phi \to \Phi$ is a continuous linear operator,

e) $\lim_{t \to t_0} T(s,t)\phi = T(s,t_0)\phi$ in the Φ-topology for each s fixed and $0 < s < t_0$, $\phi \in \Phi$, and $\lim_{s \to s_0} T(s,t)\phi = T(s_0,t)\phi$ for each t fixed and $0 < s_0 < t$, $\phi \in \Phi$,

d) for each $T > 0$ and $n > 0$

$$\sup_{0 < s < t < T} \|T(s,t)\phi\|_n < \infty \quad \forall \quad \phi \in \Phi.$$

<u>Assumptions on</u> P_t :

e) For each $t > 0$ $P_t: \Phi \to \Phi$ is a continuous linear operator

f) There exists a sequence of seminorms $\{|||\cdot|||_n;\, n > 0\}$ on Φ generating an equivalent topology as that given by the Hilbertian norms $\{\|\cdot\|_n;\, n > 0\}$ such that the following holds:

i) for each $T > 0$ there exists $m_T > 0$ s.t. for $m > m_T$ P_t has a continuous linear extension to $\Phi_{|m|}$ ($|||\cdot|||_m$-completion of Φ) and the map $s \to P_s\phi$ is $\Phi_{|m|}$-continuous,

ii) for each $T > 0$ there exists $m_T > 0$ s.t. for $m > m_T$ $\exists$ $K(m,T) > 0$ and

$$\sup_{0 \le s \le t \le T} \| P_s T(s,t)\phi \|_m \le K(m,T) \| \phi \|_m \quad \forall \ \phi \in \Phi$$

Observe that condition f (ii) above can be obtained using f (i) if we assume that for each $T > 0$ and $m \geq 0$

$$\sup_{0 \le s \le t \le T} \| T(s,t)\phi \|_m \le D(m,T) \| \phi \|_m \quad \forall \ \phi \in \Phi$$

for some $D(m,T) > 0$.

Definition 5.1 We say that the SDE (I) has a Φ'-valued solution $\xi = (\xi_t)_{t \geq 0}$ if the following four conditions hold:

i) (ξ_t) is $\mathcal{F}_t$-adapted and Φ'-valued.

ii) $\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s\phi]ds + \int_0^t \xi_s[P_s\phi]ds + W_t[\phi] \quad \forall \ \phi \in \Phi$ a.s. $\forall t \geq 0$.

iii) $\xi_\cdot \in C(R_+;\Phi')$ a.s.

iv) For each $T > 0$

$$E(\sup_{0 \le t \le T} |\xi_t[\phi]|^2) < \infty.$$

Proposition 5.1 If $(\xi_t)_{t \geq 0}$ is a solution of the SDE (I) then for each $T > 0$ there exist $q_T > 0$ and a version of ξ (denoted also by ξ) such that

$$\xi_\cdot^T \in C([0,T];\Phi'_{q_T}) \quad \text{a.s.}$$

and

$$\xi_t[\phi] = \gamma[\phi] + \int_0^t \xi_s[A_s \phi]ds + \int_0^t \xi_s[P_s \phi]ds + W_t[\phi]$$

$$\forall \phi \in \Phi, \quad 0 \le t \le T \quad \text{a.s.} \qquad (5.1)$$

The proof of this proposition is done in a manner very similar to Proposition 3.1.

Remark 1. Condition (iv) in Defintion 5.1 is implied by the following condition: For each $T > 0$

$$E\int_0^T (\xi_s[A_s \phi])^2 ds + E\int_0^T (\xi_s[P_s \phi])^2 ds < \infty \quad \forall \quad \phi \in \Phi.$$

Theorem 5.1 Let γ and $W = (W_t)_{t\geq 0}$ be as in the beginning of this lecture and suppose that A_t and P_t satisfy the assumptions (a)-(d) and (e)-(f) respectively. Then the SDE (I) has a unique solution $\xi = (\xi_t)_{t\geq 0}$ such that for each $T > 0$ there exists $p_T > 0$ and

$$\xi_\cdot^T \in C([0,T];\Phi'_{p_T}) \quad \text{a.s.}$$

and

$$E(\sup_{0\le t\le T} \|\xi\|^2_{-p_T}) < \infty.$$

Moreover ξ_t is a Φ'-valued gaussian process.

Proof. The idea is first to show that the unperturbed SDE

$$\begin{aligned} d\eta_t &= A'_t \eta_t dt + dW_t \\ \eta_0 &= \gamma \end{aligned} \qquad (II)$$

has a unique solution. This is done a manner entirely similar to the argument in Theorem 3.1 using Lemma 5.1 below instead of Lemma 3.1, and using only the assumptions on A_t. The solution is given by

$$\eta_t[\phi] = \gamma[T(0,t)\phi] + \int_0^t W_s[A_s T(s,t)\phi]ds + W_t[\phi] \ . \tag{5.2}$$

The next step is to show, by the method of successive approximations, that the stochsatic equation

$$\xi_t = \int_0^t T'(s,t)P_t' \ \xi_s ds + \eta_t \tag{5.3}$$

has a unique solution on $C(R_+;\Phi')$: The process η_t given by (5.2) has the property that for each $T > 0$ there is a $q_T > 0$ s.t. $E(C_T)^2 < \infty$ where

$$C_T(\omega) := \sup_{0 \le t \le T} \|\eta_t(\omega)\|_{-q_T} \tag{5.4}$$

Next, using assumption (f) there exist constants $c_i = c_i(T,q_T)$ $i = 1,2$ $m_T > 0$ and $n_T > 0$ s.t.

$$\|\phi\|_{q_T} \le c_1 \|\phi\|_{m_T} \le c_2 \|\phi\|_{n_T} \quad \forall \quad \phi \in \Phi \ . \tag{5.5}$$

From (f) (ii) it then follows that

$$\sup_{0 \le t \le T} \|P_s T(s,t)\phi\|_{m_T} \le K_T \|\phi\|_{m_T} \quad \forall \ \phi \in \Phi \ . \tag{5.6}$$

Let

$$\Omega_1 = \{\omega: C_T(\omega) < \infty\}$$

then $P(\Omega_1) = 1$. For $\omega \in \Omega_1$ and $0 \le t \le T$ define the sequence of successive approximations

$$\xi_t^0(\omega) = \eta_t(\omega)$$

$$\xi_t^1(\omega) = \int_0^t T'(s,t)P_s'\xi_s^0(\omega)ds + \eta_t(\omega)$$

$$\vdots$$

$$\xi_t^n(\omega) = \int_0^t T'(s,t)P_s'\xi_s^{n-1}(\omega)ds + \eta_t(\omega).$$

Then for each $n > 0$ and $\phi \in \Phi$

$$\xi_t^0[\phi] = \eta_t[\phi]$$

$$\xi_t^1[\phi] = \int_0^t \eta_s[P_sT(s,t)\phi]ds + \eta_t[\phi]$$

$$\vdots$$

$$\xi^n[\phi] = \int_0^t \xi_t^{n-1}[P_sT(s,t)\phi]ds + \eta_t[\phi]$$

$$= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-2}} \eta_{s_{n-1}}[P_{s_{n-1}}T(s_{n-1},s_{n-2})\cdots$$

$$P_{s_1}T(s_1,t)\phi]ds_{n-1} \cdots ds_1$$

$$+ \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-3}} \eta_{s_{n-2}}[P_{s_{n-2}}T(s_{n-2},s_{n-3})\cdots$$

$$P_{s_1}T(s_1,t)\phi]ds_{n-2}\cdots ds_1$$

$$+ \cdots + \eta_t[\phi].$$

Hence, using (5.5) and (5.6) one shows that the above integrals are well defined and moreover

$$|\xi_t^n(\omega)[\phi]| \le C_T(\omega)c_1c_2\left(\sum_{k=0}^{n} \frac{(K_T T)^k}{k!}\right) \|\phi\|_{n_T} \quad \forall \ \phi \in \Phi \tag{5.7}$$

and

$$|\xi^n(\omega)[\phi] - \xi^m(\omega)[\phi]| \le C_T(\omega)C_1C_2\left(\sum_{k=m+1}^{n} \frac{(K_T T)^n}{k!}\right) \|\phi\|_{n_T} \quad \forall \ \phi \in \Phi . \tag{5.8}$$

Then for each $\omega \in \Omega_1$, $0 \le t \le T$ and $\phi \in \Phi$ $\xi_t^n(\omega)$ is a Φ'-valued element for $n \ge 1$, $\xi_t(\omega)[\phi] = \lim \xi_t^n(\omega)[\phi]$ exists and

$$\sup_{0 \le t \le T} |\xi_t(\omega)[\phi]| \le C_T(\omega)c_1c_2 e^{K_T T} \|\phi\|_{n_T} \ \forall \ \phi \in \Phi . \tag{5.9}$$

Hence $\xi_t(\omega) \in \Phi'$. Moreover let $\ell_T > n_T$ be such that the injection map $\Phi_{\ell_T} \hookrightarrow \Phi_{q_T}$ is Hilbert-Schmidt and let $\{\phi_j\}_{j \ge 1} \subseteq \Phi$ be a CONS for Φ_{ℓ_T} with dual basis $\{\hat{\phi}_j\}_{j \ge 1}$ a CONS for Φ'_{ℓ_T}. Then from (5.9) we have

$$\sup_{0 \le t \le T} \sum_{j=1}^{\infty} |\xi_t(\omega)[\phi_j]|^2 \le C_T^2(\omega)c_1^2c_2^2 e^{2K_t T} \sum_{j=1}^{\infty} \|\phi_j\|_{n_T}^2 < \infty$$

and therefore for $0 \le t \le T$ and $\omega \in \Omega_1$

$$\tilde{\xi}_t(\omega) = \sum_{j=1}^{\infty} \xi_t(\omega)[\phi_j]\hat{\phi}_j \tag{5.10}$$

is a well defined element of Φ'_{ℓ_T} and $\tilde{\xi}_t(\omega)[\phi] = \xi_t(\omega)[\phi]$ $\forall \phi \in \Phi$. From now on we will write ξ_t instead of $\tilde{\xi}_t$.

Next we shall show that ξ_t is a solution of (5.3): From (5.7) and assumption f(ii) we have (suppressing $\omega \in \Omega_1$ in the writing)

$$|\xi^{n-1}[P_s T(s,t)\phi]| \le \|\xi_s^{n-1}\|_{-\ell_T} \|P_s T(s,t)\phi\|_{\ell_T}$$

$$\le C_T c_1 c_2 e^{TK_T} K_T \|\phi\|_{m_T} < \infty$$

$$\forall \quad \phi \in \Phi, \quad n \ge 1, \quad 0 \le s \le t \le T.$$

Then since

$$\xi_t^n(\omega)[\phi] = \int_0^t \xi_s^{n-1}(\omega)[P_s T(s,t)\phi]ds + \eta_t(\omega)[\phi], \quad \omega \in \Omega_1 ,$$

by the dominated convergence theorem

$$\xi_t[\phi] = \lim_{n\to\infty} \xi_t^n[\phi] = \lim_{n\to\infty} \int_0^t \xi_s^{n-1}[P_s T(s,t)\phi]ds + \eta_t[\phi]$$

$$= \int_0^t \xi_s[P_s T_{s,t})\phi]ds + \eta_t[\phi], \ \forall \quad \phi \in \Phi \quad 0 \le t \le T. \text{ a.s.} \quad (5.11)$$

which shows that ξ_t satisfies condition (ii) in Definition 5.1.

Next we shall show that $\xi_\cdot^T \in C([0,T];\Phi'_{p_T})$ a.s. for some $p_T > 0$. Let $t_0, t \in [0,T]$. Then using Lemma 5.1 below it is not difficult to show that for $\omega \in \Omega_1$

$$\left| \int_0^t \xi_u(\omega)[P_u T(u,t)\phi]du - \int_0^{t_0} \xi_u(\omega)[P_u T(u,t_0)\phi]du \right| \le C_T(w) D_T \|\phi\|_{r_T} |t-t_0|$$

for some $D_T > 0$ and $r_T > 0$. Hence the process

$$Z_t[\phi] := \int_0^t \xi_u[P_u T(u,t)\phi]du$$

is a continuous process in $t \in [0,T]$ for each $\phi \in \Phi$. Hence from (5.11) we have that $\xi_t[\phi]$ is also a continuous function of t. Moreover,

$$\sup_{0\le t\le T} |\xi_t(\omega)[\phi]| \le \{C_T(\omega) D_T + C_T(\omega)\} \|\phi\|_{r_T} \quad \forall \phi \in \Phi \quad \omega \in \Omega_1 . \quad (5.12)$$

Next let $p_T > r_T$ be such that the injection map $\Phi_{p_T} \hookrightarrow \Phi_{r_T}$ is Hilbert Schmidt and let $\{e_j\}_{j\ge1} \subsetneq \Phi$ be a CONS for Φ_{p_T} with dual basis $\{\hat{e}_j\}_{j\ge1}$ a CONS for

Φ'_{p_T}. Then from (5.12)

$$\sup_{0<t<T} \sum_{j=1}^{\infty} |\xi_t[e_j]|^2 < (C_T D_T + C_T)^2 \sum_{j=1}^{\infty} \|e_j\|^2_{r_T} < \infty .$$

Hence for $\omega \in \Omega_1$ define $\tilde{\xi}_t(\omega) = \sum_{j=1}^{\infty} \xi_t(\omega)[e_j]\hat{e}_j$ which is an element in Φ'_{p_T}. Then

$$\tilde{\xi}_t(\omega)[\phi] = \xi_t(\omega)[\phi] \quad \forall \quad \phi \in \Phi \quad 0 < t < T, \ \omega \in \Omega_1.$$

From now on we write ξ_t instead of $\tilde{\xi}_t$. Then by the dominated convergence theorem, since $\xi_t(\omega)[e_j]$ is continuous on t for each $j > 1$ we have

$$\lim_{t\to t_0} \|\xi_t - \xi_{t_0}\|^2_{-p_T} = \lim_{t\to t_0} \sum_{j=1}^{\infty} (\xi_t[e_j] - \xi_{t_0}[e_j])^2$$

$$= \sum_{j=1}^{\infty} \lim_{t\to t_0} (\xi_t[e_j] - \xi_{t_0}[e_j])^2 = 0 \quad t_0 \in [0,T]$$

Then $\xi^T(\omega) \in C([0,T]; \Phi'_{p_T})$ for some $p_T > 0$ and $\omega \in \Omega_1$ where $P(\Omega_1) = 1$.

Also from (5.4) and (5.12) we have that for each $T > 0$

$$E(\sup_{0<t<T} |\xi_t[\phi]|^2) < \infty \quad \forall \quad \phi \in \Phi.$$

which shows condition (iv) in Definition 5.1. Moreover from (5.12) and since $E(C_T^2) < \infty$

$$E(\sup_{0<t<T} \|\xi\|^2_{-p_T}) < E(C_T D_T + C_T)^2 \sum_{j=1}^{\infty} \|e_j\|^2_{r_T} < \infty.$$

A simlar argument to that at end of Step 3 in Theorem 3.1 gives condition (ii) in Definition 5.1, i.e.

$$\xi_{\cdot} \in C(R_+; \Phi') \quad \text{a.s.}$$

Hence, ξ is a solution of (5.3).

To show uniqueness let X be any solution of (5.3). For the present assume that X_t sastisfies the following condition:

(*) For each $T > 0$ there exists $\rho'_T > 0$ s.t. $X^T_\cdot \in C([0,T]; \Phi'_{\rho'_T})$ a.s.

W.L.O.G. let $\rho'_T > \rho_T$ and

$$\Omega_2 = \{\omega: \sup_{0 \le t \le T} \|X_t(\omega)\|_{-\rho_T} < \infty\}.$$

Then $P(\Omega_2) = 1$. Fix $\omega \in \Omega_1 \cap \Omega_2$ and let $0 \le t \le T$. Then for each $\phi \in \Phi$ (suppressing ω in the writing)

$$X_t[\phi] = \int_0^t X_s[P_s T(s,t)\phi]ds + \eta_t[\phi] .$$

Next, if ξ^n_t is the sequence of successive approximations defined prior to (5.7) we have

$$X_t[\phi] - \xi^0_t[\phi] = \int_0^t X_s[P_s T(s,t)\phi]ds \tag{5.13}$$

$$X_t[\phi] - \xi^1_t[\phi] = \int_0^t X_s[P_s T(s,t)\phi]ds - \int_0^t \xi^0_s[P_s T(s,t)\phi]ds$$

$$\vdots$$

$$X_t[\phi] - \xi^n_t[\phi] = \int_0^t (X_s[P_s T(s,t)\phi] - \xi^{n-1}_s[P_s T(s,t)\phi])ds .$$

Hence

$$X_t[\phi] - \xi^n_t[\phi] = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \{X_{s_n}[P_{s_n} T(s_n, s_{n-1}) \cdots P_{s_1} T(s_1,t)\phi$$

$$- \xi^1_{s_n}[P_{s_n} T(s_n,s_{n-1}) \cdots P_{s_1} T(s_1,t)\phi]\}ds_n \cdots ds_1$$

$$\overset{(5.13)}{=} \int_0^t \int_0^{s_1} \cdots \int_0^{s_n} X_{s_{n+1}}[P_{s_{n+1}} T(s_{n+1},s_n) \cdots$$

$$P_{s_1} T(s_1,t)\phi]ds_{n+1} \cdots ds_1$$

Then using inequalities similar to (5.5) and (5.6) it follows that

$$|X_t[\phi] - \xi_t^n[\phi]| \leq \sup_{0 \leq t \leq T} \|X_t\|_{-p_T} c_1 c_2 \frac{(K_T T)^n}{n!} \|\phi\|_m$$

for some positive constants c_1, c_2, K_T and m. Hence

$$\sup_{0 \leq t \leq T} |X_t[\phi] - \xi^n[\phi]| \to 0 \quad \text{as} \quad n \to \infty \ .$$

Thus $P(X_t = \xi_t \ \ 0 \leq t \leq T) = 1$ and a similar argument to that at the end of Step 4 in Theorem 3.1 gives that $P(X_t = \xi_t \ \ t \geq 0) = 1$.

The next step is to show that the solution $\xi_\cdot$ of (5.3) obtained above is also a solution of (I). In order to do that we need the following Lemma.

Lemma 5.1 Suppose conditions (a)-(d) on A_t hold and let B be any continuous linear operator from Φ to Φ. Then for each $F \in \Phi'$ and $0 \leq u \leq t$

a) $$F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BT(u,s)A_s\phi]ds \qquad \forall \ \phi \in \Phi$$

b) $$F[BT(u,t)\phi] = F[B\phi] + \int_u^t F[BA_sT(s,t)\phi]ds \qquad \forall \ \phi \in \Phi.$$

The proof of the Lemma is similar to that of Lemma 3.1 using the Kolmogorov Forward and Backward equations:

$$\frac{d}{ds} T(u,s)\phi = T(u,s)A_s\phi \qquad 0 \leq u \leq s, \quad \phi \in \Phi$$

$$\frac{d}{du} T(u,s) = -A_u T(u,s)\phi \qquad 0 \leq u \leq s, \quad \phi \in \Phi.$$

End of the Proof of Theorem 5.1

We shall show that ξ is also a solution of (I) and that it is unique. Let $\omega \in \Omega_1$ then from (5.11) (suppressing ω in the writing) we have

$$\xi_t[\phi] = \int_0^t \xi_s[P_s T(s,t)\phi]ds + \eta_t[\phi] \quad \forall \quad \phi \in \Phi \quad 0 < t < T . \qquad (5.14)$$

Next, applying Lemma 5.1(a) with $B = P_u$ and $F = \xi_u$ we have

$$\xi_u[P_u T(u,t)\phi] = \xi_u[P_u\phi] + \int_u^t \xi_u[P_u T(u,s)A_s\phi]ds$$

and therefore

$$\int_0^t \xi_u(\omega)[P_u T(u,t)\phi]du = \int_0^t \xi_u(\omega)[P_u\phi]du + \int_0^t\int_u^t \xi_u(\omega)[P_u T(u,s)A_s\phi]dsdu$$

$$= \int_0^t \xi_u(\omega)[P_u\phi]du + \int_0^t\int_0^s \xi_u(\omega)[P_u T(u,s)A_s\phi]duds. \qquad (5.15)$$

But from (5.14) since

$$\xi_s[A_s\phi] = \int_0^s \xi_u[P_u T(u,s)A_s\phi]du + \eta_s[A_s\phi],$$

using the above expression in the second term of (5.15) we obtain

$$\int_0^t \xi_u[P_u T(u,t)\phi]du = \int_0^t \xi_u[P_u\phi]du + \int_0^t \xi_s[A_s\phi]ds - \int_0^t \eta_s[A_s\phi]ds. \qquad (5.16)$$

But also from (5.14) we obtain

$$\int_0^t \xi_u[P_u T(u,t)\phi]du = \xi_t[\phi] - \eta_t[\phi].$$

Hence from the above expression and (5.16),

$$\xi_t[\phi] - \eta_t[\phi] = \int_0^t \xi_s[P_s\phi]ds + \int_0^t \xi_s[A_s\phi]ds - \int_0^t \eta_s[A_s\phi]ds$$

i.e.

$$\xi_t[\phi] = \int_0^t \xi_s[P_s\phi]ds + \int_0^t \xi_s[A_s\phi]ds + \eta_t[\phi] - \int_0^t \eta_s[A_s\phi]ds$$

but since $\eta_t[\phi] - \int_0^t \eta_s[A_s\phi]ds = \gamma[\phi] + W_t[\phi]$,

$$\xi_t[\phi] = \int_0^t \xi_s[P_s\phi]ds + \int_0^t \xi_s[A_s\phi]ds + \gamma[\phi] + W_t[\phi] \quad \forall \quad \phi \in \Phi,$$

i.e.
$$d\xi_t = A_t'\xi_t dt + P_t'\xi_t dt + dW_t.$$

Finally we shall show the uniqueness of the solution (I) proving that any other solution $\bar{\xi}_t$ of (I) satisfies the SDE (5.3). Let Ω_3 be the set of probability one given by Proposition 5.1 s.t.

$$\bar{\xi}_\cdot^T \in C([0,T];\Phi'_{q_T}) \quad \text{a.s.}$$

and

$$\bar{\xi}_t[\phi] = \gamma[\phi] + \int_0^t \bar{\xi}_s[A_s\phi]ds + \int_0^t \bar{\xi}_s[P_s\phi]ds + W_t[\phi]$$

$$\forall \quad \phi \in \Phi \quad 0 < t < T \quad \text{a.s.} \tag{5.17}$$

WLOG we can take $q_T > p_T$. Let $\omega \in \Omega_3 \cap \Omega_1$. Then (suppressing ω in the following) from (5.17) we have that for $0 < s < T$

$$W_s[A_sT(s,t)\phi] = \bar{\xi}_s[A_sT(s,t)\phi] - \int_0^t \bar{\xi}_u[P_uA_sT(s,t)\phi]du -$$
$$- \int_0^t \bar{\xi}_u[A_uA_sT(s,t)\phi]du - \gamma[A_sT(s,t)\phi]. \tag{5.18}$$

On the other hand from (5.3) and (5.2) if $0 < t < T$

$$\xi_t[\phi] - \int_0^t \xi_s[P_sT(s,t)\phi]ds = \int_0^t W_s[A_sT(s,t)\phi]ds + \gamma[T(0,t)\phi] + W_t[\phi]. \tag{5.19}$$

Hence using (5.18) in (5.19) we have

$$\xi_t[\phi] - \int_0^t \xi_s[P_sT(s,t)\phi]ds = \int_0^t \bar{\xi}_s[A_sT(s,t)\phi]ds$$

$$- \int_0^t\int_0^s \bar{\xi}_u[P_uA_sT(s,t)\phi]duds - \int_0^t\int_0^s \bar{\xi}_u[A_uA_sT(s,t)\phi]duds$$

$$- \int_0^t \gamma[A_sT(s,t)\phi]ds + \gamma[T(0,t)\phi] + W_t[\phi] \quad . \qquad (5.20)$$

Next, using Lemma 5.1(b) with $F = \gamma$, $B = I$ we have

$$- \int_0^t \gamma[A_sT(s,t)\phi]ds + \gamma[T(0,t)\phi] = \gamma[\phi]. \qquad (5.21)$$

Again, applying Lemma 5.1(b) with $F = \bar{\xi}_u$, $B = P_u$ and with $F = \bar{\xi}_u$ and $B = A_u$ we have the following two expressions

$$-\int_0^t \bar{\xi}_u[P_uA_sT(s,t)\phi]ds = \bar{\xi}_u[P_u\phi] - \bar{\xi}_u[P_uT(u,t)\phi] \qquad (5.22)$$

$$-\int_0^t \bar{\xi}_u[A_uA_sT(s,t)\phi]ds = \bar{\xi}_u[A_u\phi] - \bar{\xi}_u[A_uT(u,t)\phi]. \qquad (5.23)$$

Hence, using (5.21), (5.22) and (5.23) in (5.20) we have

$$\xi_t[\phi] - \int_0^t \xi_s[P_sT(s,t)\phi]ds = \int_0^t \bar{\xi}_s[A_sT(s,t)\phi]ds + \int_0^t \bar{\xi}_u[P_u\phi]du$$

$$- \int_0^t \bar{\xi}_u[P_uT(u,t)\phi]du + \int_0^t \bar{\xi}_u[A_u\phi]du - \int_0^t \bar{\xi}_u[A_uT(u,t)\phi]du$$

$$+ \gamma[\phi] + W_t[\phi],$$

that is,

$$\xi_t[\phi] - \int_0^t \xi_s[P_sT(s,t)\phi]ds = \int_0^t \bar{\xi}_u[P_u\phi]du + \int_0^t \bar{\xi}_u[A_u\phi]du$$

$$+ \gamma[\phi] + W_t[\phi] - \int_0^t \bar{\xi}_u[P_uT(u,t)\phi]du.$$

Then using (5.17)

$$\bar{\xi}_t[\phi] - \int_0^t \bar{\xi}_s[P_s T(s,t)\phi]ds = \xi_t[\phi] - \int_0^t \xi_s[P_s T(s,t)\phi]ds = \eta_t[\phi],$$

i.e. $\bar{\xi}_t$ satisfies (5.3). The uniqueness now follows from the uniqueness of the solution of the SDE (5.3) using Proposition 5.1, which gives condition (*).

Finally the Gaussian property of the solution ξ follows from the fact that for each $\phi \in \Phi$ $\xi_t[\phi]$ is the a.s. limit of a sequence (the successive approximations) of gaussian random variables. Q.E.D.

Special case. Let $A = -L$ be the infinitesimal generator of a one parameter semigroup as in Lecture 3 and consider the SDE

$$d\xi_t = -L'\xi_t dt + P_t'\xi_t dt + dW_t$$

$$\xi_0 = \gamma$$

The unperturbed equation is a model used in neurophysiological applications (Example 3.1 and Lecture 4). However it is important to observe that in this field the kind of perturbation that occur are more likely to be nonlinear rather than linear.

Example 5.1. This example occurs in fluctuation theorems for interacting particle diffusions and has been considered by McKean [9], Hitsuda and Mitoma [4] and Mitoma [12].

For $n > 1$ let

$$Y^{(n)}(t) = (Y_1^{(n)}(t),\ldots,Y_n^{(n)}(t))$$

be an n-particle diffusion given by the SDE

$$Y_k^n(t) = Y_k + \frac{1}{n}\sum_{j=1}^{n}\int_0^t a(Y_k^{(n)}(s),Y_j^{(n)}(s))dB_k(s)$$

$$+ \frac{1}{n}\sum_{j=1}^{n}\int_0^t b(Y_k^{(n)}(s),Y^{(n)}(s))ds \qquad (k = 1,\dots,n)$$

where $(Y_k,B_k)_{k\geq 1}$ are independent copies of (Y,B) and Y is a random variable independent of the real valued Brownian motion B_t. The coefficients $a(x,y)$ and $b(x,y)$ are bounded C^∞-functions in (x,y).

For each $t > 0$ consider

$$U^{(n)}(t) = \frac{1}{n}\sum_{j=1}^{n}\delta_{Y_j^{(n)}(t)}$$

where δ_x is the unit mass at x. $U^{(n)}(t)$ is a measure valued process. McKean [9] has shown that

$$U^{(n)}(t) \overset{a.s.}{\to} u(dx,t)$$

where $u(dx,t)$ is the probability distribution of $X(t)$ that satisfies

$$dX(t) = \alpha(X(t),t)dB_t + \beta(X(t),t)dt$$

where

$$\alpha(x,t) := \int_R a(x,y)u(dy,t)$$

$$\beta(x,t) := \int_R b(x,y)u(dy,t).$$

Moreover, McKean [9] has also shown that $u(dy,t)$ has a density $u(x,t)$ and that $\alpha(x,t)$, $\beta(x,t)$ and $u(x,t)$ are C^∞-functions in $R \times R_+$.

Let

$$S_n(t) = \sqrt{n}\,[U^{(n)}(t) - u(\cdot,t)].$$

Hitsuda and Mitoma [4] have shown that the measure valued processes $S_n(\cdot)$ converge weakly to the solution $\xi = (\xi_t)$ of the stochastic evolution equation

$$d\xi_t = A_t'\xi_t dt + P_t'\xi_t dt + dW_t \tag{5.24}$$

where for $\phi \in \Phi$

$$(A_t\phi)(x) = \frac{1}{2}\alpha(x,t)^2\phi^{(2)}(x) + \beta(x,t)\phi^{(1)}(x) \tag{5.25}$$

$$(P_t\phi)(x) = \int_R b(y,x)\phi^{(1)}(y)u(y,t)dy$$

$$+ \int_R \alpha(y,t)a(x,y)\phi^{(2)}(y)u(y,t)dy \tag{5.26}$$

and W_t is a zero mean Φ'-valued gaussian process with independent increments, $W_0 = 0$. As pointed out in Mitoma [12], the nuclear space appropriate to the problem is given by the space Φ of real valued functions ϕ such that $\phi \in \Phi$ iff $\psi(x)\phi(x) \in \mathcal{S}$ where

$$\psi(x) = \int_R e^{-|z|}\rho(x - z)dz$$

and ρ is the usual mollifier

$$\rho(x) = \begin{cases} c\cdot \exp(1/(1 - |x|^2)) & |x| < 1 \\ 0 & |x| > 1. \end{cases}$$

Notice that Φ is a modification of $\mathcal{S}$ with the following relations among the norms defining their corresponding topologies:

$$|||\phi|||_{n,\Phi} = \|\psi\phi\|_{n,\mathcal{S}} \tag{5.27}$$

$$\|\phi\|_{n,\Phi} = \|\psi\phi\|_{n,\mathcal{S}} \tag{5.28}$$

where

$$|||f|||_{n,\mathcal{S}} = \sup_{0 \leq jk \leq n} \sup_{x \in R} |D^k f(x)| \qquad n \geq 1 \tag{5.29}$$

$$\|f\|^2_{n,\mathcal{S}} = \sum_{k=0}^{n} \int_R (1 + x^2)^{2n} |D^k f(x)|^2 dx \quad n \geq 1. \tag{5.30}$$

It can be shown (see [12]) that under the above conditions the SDE (5.24) satisfies the assumptions of Theorem 5.1 and therefore $S_n(\cdot)$ converge weakly to the unique solution of the stochastic evolution equation (5.24).

The example just discussed is an instance where the two parameter evolution semigroup $T(s,t)$, its generator A_t ad the perturbator P_t can all be defined directly on a countably Hilbertian nuclear space Φ so as to satisfy the assumptions stated at the beginning of this lecture. It is worth noting that, in many cases, these operators may be more naturally defined on a Hilbert or Banach space, as e.g., in the work of Kato and Tanabe [8, 14]. In such cases the problem of finding a Φ for which the assumptions concerning A_t and P_t are valid, has to be solved first before the results of this lecture can be applied.

2. Φ'-VALUED DIFFUSION STOCHASTIC DIFFERENTIAL EQUATIONS

A more general Φ'-valued SDE is given by

$$\begin{aligned} d\xi_t &= A(t,\xi_t)dt + B(t,\xi_t)dW_t \\ \xi_0 &= \gamma \end{aligned} \tag{5.31}$$

where the coefficient functions A, B are of the following type

a) $A\colon\ R_+ \times \Phi' \to \Phi'$

b) $B: R_+ \times \Phi' \to \mathcal{L}(\Phi',\Phi')$

where $\mathcal{L}(\Phi',\Phi')$ denotes the class of all linear continuous operators from Φ' to Φ', γ is a Φ'-valued random variable and W_t is a Φ'-valued Wiener process with covariance Q. In this case it is necessary to define a stochastic integral of the type

$$\int_0^t f_s(w)dW_s(w)$$

where $f_t(\omega) \varepsilon \mathcal{L}(\Phi',\Phi')$.

The following are conditions under which a unique solution to (5.31) exits:

For each $T > 0$ and sufficiently large $m > 0$, there exist numbers $r > 2$, $\theta > 0$ and $p > m$ such that A, B, the initial measure μ_0 for ξ_0 and the covariance functional Q satisfy the following properties

(IC) - <u>Initial Condition</u>

$$\int_{\Phi'} \|u\|_{-m}^{r} \mu_0(du) < \infty$$

(CC) <u>Coercivity</u>: Let j_m be the canonical isomorphism between Φ_m and Φ_m'. For each $t < T$ and $u \varepsilon \Phi$

$$2A_t(j_m u)[u] + (r - 1)\|Q_{B_t^*(u)}\|_{-m,-m} \leq \theta(1 + \|j_m(u)\|_{-m}^2)$$

$\|\cdots\|_{-m,-m}$ is the trace norm of the nuclear operator determined by the bilinear form $Q_{B^*(u)}$ where

$$Q_{B_t^*(u)}(\phi,\psi) = Q(B_t^*(u)\phi, B_t^*(u)\psi) \qquad \phi,\psi \varepsilon \Phi .$$

and, $B_t^*(u) \varepsilon \mathcal{L}(\Phi, \Phi)$ is the adjoint operator defined by the relation.

$B_t(u)\, f[\phi] = f[B_t^*(u)\phi]$ for all $f \varepsilon \Phi'$ and all $\phi \varepsilon \Phi$

(LG) Linear Growth for each $t < T$ and $u \in \Phi'_m$

$$A_t(u) \in \Phi'_p$$

and

$$\|A_t(u)\|^2_{-p} < \theta(1 + \|u\|^2_{-m})$$

$$\|Q_{B_t^*(u)}\|^2_{-m,-m} < \frac{1}{2}\,\theta^{2/r}(1 + \|u\|^2_{-m})$$

(MC) Monotonicity. For ecah $t < T$ and $u,v \in \Phi'_m$

$$A_t(u), A_t(v) \in \Phi'_p$$

and

$$2(u - v, A_t(u) - A_t(v))_{-p} + (r - 1)\|Q_{B_t^*(u) - B_t^*(v)}\|_{-p,-p}$$

$$< \theta\|u - v\|^2_{-p}$$

(JC) Joint Continuity

$$A: \ [0,T] \times \Phi' \to \Phi'$$

and

$$B: \ [0,T] \times \Phi' \to \mathcal{L}(\Phi', \Phi')$$

are each jointly continuous.

The proof of the existence of the solution will appear in a forthcoming paper by G. Kallianpur and R. Wolpert.

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